PSYCHOLOGICAL ANALYSIS OF THE FUNDAMENTALS OF ARITHMETIC

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COUNTING AS AN INDIVIDUAL ABILITY AND AS A RACIAL MODE OF DEALING WITH EXPERIENCES

INTRODUCTION

The investigations reported in this monograph deal with the mental processes of adults and children when they are counting or making the simpler number combinations. The reader can gain a general understanding of some of the methods employed and of the type of psychological conclusion reached if he will try a few simple experiments which require no more elaborate equipment than a watch, a pencil, a sheet of paper, and a book.

MEASURING RATES OF COUNTING ALOUD

The first experiment consists in counting aloud as fast as possible for ten seconds, using the number names from "one" to "ten" inclusive and repeating the series as often as possible. The result will be somewhere between six and nine series, depending on the fluency of the individual. The second experiment is like the first except that the number names from "one" to "twenty" inclusive are used. The counting which involves the articulation of the longer words—"eleven," "twelve," "thirteen," etc.—will cover in the aggregate from 25 to 35 per cent fewer numbers than the counting which is limited to the use of the number names from "one" to "ten."

COUNTING SILENTLY

The third and fourth experiments are like the first and second except that the counting is done silently. The results when one counts silently are in most instances not very different from the results secured in counting aloud. There are a few persons who are slower in silent counting than in counting aloud. A few are faster. Whatever the variations in rate of silent counting from the rate of counting aloud, the relation of the aggregate number covered when counting up to "ten" to the aggregate number covered when counting up to "twenty" is the same for counting silently as for counting aloud. Counting up to "twenty" is accordingly shown to be slower by an appreciable percentage in all cases.

ALL COUNTING INVOLVES THE USE OF NUMBER NAMES

The fact that the rate of counting up to "twenty" is slow as contrasted with the rate of counting up to "ten" throws light on the psychology of all

counting The slower rate of counting up to "twenty" makes it clear that counting is a mental process which uses number names and depends for its rate on the character of the number names employed The mental process of counting up to "twenty" is slow under all conditions because the number names from "eleven" to "twenty" are elaborate Number names are the devices or means which the mind has to use in performing its processes of enumerating Especially significant is the fact that the psychological process involved in counting is not freed from its dependence on number names even when one counts silently Most observers note that there is a strong disposition during silent counting to make movements with the tongue and with the vocal apparatus Some persons, indeed, are so disturbed by the demand that they suppress articulation that they count more slowly silently than aloud and sometimes become entirely confused, especially between "eleven" and "twenty"

COUNTING BEYOND "TWENTY"

Similar experiments can be tried to determine the rate of counting from "twenty-one" to "thirty," from "thirty-one" to "forty," and so on It will be found that there are three distinct methods of counting in these sections of the number system Some persons give full emphasis to the "twenty" part of each of the number names from "twenty one" to "twenty nine" In both oral and silent counting, they use the full words "twenty-one," "twenty two," and so on A second group of persons slur the "twenty" and lay emphasis on the "one," "two," "three" A third class of persons omit altogether the "twenty" after the first use of the word to mark the transition from "nineteen" They say "twenty, one, two, three," and so on, until they reach "thirty" Those who belong to the first group, that is, those who use the "twenty" with full emphasis, are slow Those who omit the "twenty," "thirty," "forty," and so on are rapid, but, in counting long series, they are likely to make mistakes, they sometimes repeat a decade or omit a decade The persons who have learned to slur the name of the particular decade are intermediate in speed and, as contrasted with the very rapid group, are sure of their position in long series

RELATION OF TAPPING AND COUNTING

The reader who has tried the experiments thus far described can now carry the analysis of his counting somewhat farther Let him take a pencil and a sheet of paper and, with his watch before him, make as many marks as he can by tapping on the paper for ten seconds On counting the marks, he will find a very striking similarity between the number of taps made

and the aggregate number which he covered when he counted up to "ten" for ten seconds In other words, the rate at which one can send motor impulses to the hand is very closely related to the rate at which one can send simple motor impulses to the vocal cords There is evidently a close relation between the two kinds of action A further important fact to note in this connection is that both making marks and counting by using the number names from "one" to "ten" are among the most rapid reactions of which an individual is capable If one tries to tap with the foot or to nod the head as fast as possible, one finds that the rate is slower than that attained in counting or in making marks Counting by means of the number names from "one" to "ten" is thus shown to be a very fluent process

MATCHING THE NUMBER SERIES TO VISUAL OBJECTS

Let the experimenter now change conditions once more This time the effort is to discover how rapidly one can count a series of objects The most available series of objects for convenient testing outside the laboratory will be found in the letters on a printed page The experiment requires that the eyes be fixed on the page and not on the individual's watch A convenient method of procedure is as follows Let the experimenter place his pencil on a letter at the beginning of a line and then look at his watch until the second hand reaches one of the major points on the dial Let him now look away from the watch to the book and count letters as fast as he can Let him count until he has covered as many letters as he covered numbers when he counted aloud from "one" to "ten" for ten seconds In order to make the experiment strictly comparable with the first experiment, the experimenter should use only the number names from "one" to "ten" in counting When he has covered from sixty to ninety letters, as the case may require, let the experimenter look at his watch He will find that he consumed from twenty to twenty five seconds in counting as many letters as he repeated number names from "one" to "ten" in ten seconds Evidently the application of number names to a particular series of objects involves something more than the mere ability to say the number names There must be some discrimination of the objects counted and there must be an application of the separate number names to the individual objects discriminated

Counting visual objects is thus seen to be a complex process The person who counts has within himself an established series of reactions which he has so fully mastered that he can execute them at his maximum physiological rate When he attempts to apply his developed reactions to objects which he sees with his eyes, he cannot react with the speed that is

possible when he is merely repeating the number names silently Counting things is a more complex process than repeating the names of the numbers

INDIVIDUAL DIFFERENCES IN COUNTING

If the reader can extend his inquiry by experimenting on a number of persons he will find some further interesting facts He will find that there are marked individual differences in all the processes under discussion Differences in rate will be especially conspicuous if the experiments are tried on both adults and children If certain children below the age of ten are observed the rate of counting will be found to be very slow The relative complexity of the different kinds of counting will however be found to be the same in spite of the individual differences which appear

COUNTING AS AN ACTIVE PROCESS

On the basis of the facts presented several psychological generalizations are justified First counting is an active process The individual who counts is not a passive recipient of impressions impressions however strong in themselves do not bring the number series to the mind The number series is something which the individual imposes on the world by his active responses Second it is evident that the active process involved in counting is related to the individual s nervous organization and cannot take place any faster than the nervous system can send out motor impulses Third the differences between the rates of various kinds of counting which are exhibited regularly by children and adults lead to the conclusion that the active process of counting is one which matures with age and experience Fourth the evidence shows that the rate of counting depends on the complexity of the reactions involved When the reactions are complex as in the use of the number names above ten, the rate is slow Finally, evidence has been presented which shows that the application of the active process of counting to perceptual objects such as letters on a printed page involves processes other than the mere repetition of the number series

TYPES OF EVIDENCE OTHER THAN THAT DERIVED FROM EXPERIMENTATION

In addition to the foregoing experimental evidence there are two groups of facts which the reader should have in mind as relevant to the investigations to be reported in this monograph The first series of facts is derived from the experience of teachers who give courses in arithmetic and observe the difficulties which pupils encounter The second series of facts is contributed by social psychology and is derived from an account

of the struggles through which the race has passed and of the results which have been achieved in the course of the perfection of the number system

NUMEROUS FAILURES IN SCHOOL COURSES IN ARITHMETIC

Teachers find that more pupils fail in arithmetic than in any other elementary subject Pupils who are altogether normal in intelligence sometimes show a lack of mastery of number This fact has led to the belief that the understanding of number depends on the inheritance of a special kind of ability There are not a few mature persons who say frankly that they can do nothing with number These same people can read which proves that they are not lacking in general intelligence or in ability to use symbols In like fashion, it has been shown in school surveys[1] that in many cases pupils who can read fail to satisfy the requirements in arithmetic The fact that pupils find number less easy to master than reading furnishes a very strong motive for intensive psychological study of number ideas

Not only is the learning of arithmetic found to be difficult, but it is found in the course of school experience that the difficulties in this subject are cumulative If a pupil in one of the lower grades fails to understand some phase of the number work he will become increasingly confused In this sense, arithmetic is a more sequential school subject than any other One can omit a section of geography or history and yet be fairly intelligent in other sections of the subject one can fail to learn certain individual words and yet be competent in dealing with the ordinary vocabulary used in school textbooks In arithmetic the pupil is confronted by a system of experiences which hold together in a unique way The inner coherence of the sections of arithmetic is one of its virtues It makes possible a type of sequential teaching and satisfactory review impossible in other subjects These same characteristics however make arithmetic the despair of anyone who has failed to understand any of the essential early steps in the subject

DIFFICULTIES IN ARITHMETIC OFTEN SUBJECTIVE AND OBSCURE

The difficulties which pupils encounter in understanding number facts are often unrecognized by teachers[2] A large part of the pupil s activity in solving a problem in arithmetic is not open to external observation

[1] Charles Hubbard Judd *Measuring the Work of the Public Schools* pp 26–29 New York Russell Sage Foundation 1916

[2] G T Buswell with the co-operation of Lenore John *Diagnostic Studies in Arithmetic* pp 1 3 Chicago Department of Education University of Chicago 1926

PRIMITIVE NUMBER SYSTEMS LIMITED

What we know about primitive number systems as revealed through a study of practices among semi civilized peoples of the present day and through an analysis of the history of the number names makes it clear that the first discrimination of things by human minds was limited to the recognition of small groups of objects At first there were no devices such as those which number systems supply to modern men for dealing with groups of things The precise comprehension of many things and a definition of their number or of the magnitude of any group could not take place without the evolution of complicated devices for enumerating and recording A striking example of the need of some method and means of thinking with precision is found in the fact that even in our own day the Apache Indians of Arizona keep a record of their ponies by carrying a little sack filled with pebbles the number of which corresponds to the number of ponies in their drove If an Indian loses his counters he is very much confused He has no other means of determining with exactness the extent of his possessions

NUMBER AND RANGE OF ATTENTION

The fact that the mind must have some device for arranging and recording its experiences of many things is explained by the limitations of human attention The range of objects which can be recognized in a single field of attention is very limited An observer can discriminate three or four objects with definiteness but he becomes confused if he is confronted by a group of twelve or fifteen objects In order to hold fifteen objects in mind an observer must subdivide the large group or use some other device of simplification He must also be able to record in some way the various steps which he takes in subdividing the large group

The race struggled for a long time in solving the problem which was created by the limited attention of the human mind on the one hand and by the desire to gain precise knowledge of large groups of possessions on the other The Latin word *calculus* means a pebble and our English derivatives from this word such as the verb calculate furnish philological evidence of the fact that the Romans like the Apache Indians helped themselves in gaining precise knowledge of large groups of objects by using pebbles

FINGERS AS CONVENIENT TALLIES

The tallies which men of all ages have found most readily accessible as aids in the analysis of large groups of objects are the fingers Our decimal system and the systems which use 5 or 20 as basal numbers are

clear indications that the fingers and toes were early adopted by man as aids to his limited range of attention

NUMBER AS A DEVICE FOR ORDERLY THINKING

If we recognize number as an aid to the discriminative recognition and grouping of objects, we shall understand one of the facts which was observed in the experiments with which this chapter opened It was there discovered that counting is an active process The observer counts not because the outer world impresses number upon his senses but because he is trying to arrange his experiences in such a way that he can comprehend them Number is a device to aid thinking, it is an aid to the mind It is a human invention, not a fact of the natural world

PRIMITIVE NUMBER NAMES BASED ON ANALOGIES

In the course of human effort to evolve suitable devices to aid precise thinking in the presence of large groups of objects, a great many experiments have been tried For example, when men first began to look for devices to aid their thinking, they compared unfamiliar groups with well-known groups Thus, one writer on primitive number words records the use of the phrase "toes of the ostrich" by a primitive tribe for "four" The primitive people who used this phrase were evidently helping themselves by a form of expression which, fully stated, would be something like this "I have as many of these as an ostrich has toes" Similar references to familiar objects are found in many languages which use as number names the names of the fingers and hand.

So long as men depended on these analogies or concrete pictures to support their thinking, they could not carry large groups of objects in mind The range of objects which they could observe with precision was limited to twenty at most Usually, it was limited to ten or even less We may call the method of counting described the method of using "concrete number"

MATURE NUMBER NAMES NOT CONCRETE

The stage of concrete number was passed when the number names began to be familiar enough so that the pictures in which they originated faded away and the number names began to have fixed meanings

A series of facts borrowed from modern language may help to make clear how this important psychological step was taken We have need in modern life of an elaborate series of words which will help in the discrimination of colors Accordingly, we have added to the few ancient words, such as "red," "green," "yellow," and "blue," which help us to make the

grosser distinctions, a host of very concrete modern names, such as "orange," "violet," "indigo blue," "powder blue," "navy blue," and "maroon" Such color names as "orange" derived their meaning from mental pictures of familiar objects Through frequent use and increasing familiarity, the imagery gradually became less and less pronounced, and the color ideas came to be more and more independent. The word "indigo," which originated as the name of a plant but which is not thought of by most persons except as the name of a color is a suggestive example of the transition from the concrete reference to an object to the purely abstract reference to a color quality. Such color names as "powder blue" are derived from even more subtle references to concrete experiences.

SERIAL CHARACTER OF NUMBER NAMES

In describing the quantitative aspect of groups of objects, men gradually evolved a series of words which came to have meanings quite independent of the objects from the names of which they were originally derived The new meanings were made possible by the serial relation in which the number names finally became fixed in thought Such a word as "five" derives its meaning in the modern number series not by recalling some outside concrete objects but by taking its place in the number series For a full understanding of its meaning, the word "five" requires a knowledge of the series of ideas to which this word belongs One must know "four" and "six" in order to apprehend fully the meaning of "five"

Not only did the series of number names thus detach themselves from the objects from which they were derived, but they ultimately developed into a succession of repeated cycles which carried the mind far beyond its early concrete experiences When men learned that they could repeat the nine fundamental number names in the combinations "one and twenty," "two and twenty," and so on, and when they learned to form the compounds "one hundred," "two hundred," and the like, they transcended completely the concrete numbers with which their thinking began

SUBSTITUTION OF THE NUMBER NAMES FOR TALLIES

When fully established, the series of number names became a subjective substitute for the tallies which were originally used for recording large numbers A man possessed of a fixed series of number names can repeat these names one after the other until he has matched a name to each object He thus substitutes the repetition of number names for the use of pebbles or fingers or concrete images When he counts a long series of objects, he has only to remember the number name which he used at

the end of the process Because of the character of the number series, this number name represents the series as a whole at the same time that it serves as the device for counting the last object

The counting series was soon found to be useful not only in enumerating objects but also in making combinations The names 'one and twenty,' 'two and twenty,' and the like suggest combinations Such names as 'twenty,' "thirty,' and 'forty' suggest combinations of groups There is in these cases no suggestion of combinations of groups like 'seven and 'five' In other words only the simplest and the most regular combinations are thought of at first The idea of combination as distinct from the idea of mere enumeration required a long period of intellectual evolution

EARLY NUMBER SYSTEMS SUBJECTS OF SPECULATIVE INTEREST

The discovery of number as a series of independent ideas full of implications was a source of great intellectual stimulation to the ancient thinkers who first began to speculate about the meaning of these ideas Greek history tells of the speculations which were common among the Greek philosophers who were engrossed in finding out the properties of numbers The philosophers were captivated by the fact that some numbers are related to two and are what we call today 'even numbers while other numbers are of a wholly different type and are what we call 'odd' They attached all kinds of fantastic ideas to these different types of numbers Numbers were thought of as male and female they were made the basis of religious speculations In the course of these speculations the laws of combination were discovered which constitute what we know today as the "science of number' Speculation in this case, as in many other cases, led to the formulation of modern science

FIXITY OF ORDER ESSENTIAL TO THE NUMBER SERIES

The fact that the number series must maintain a fixed order was early recognized as a fundamental necessity As pointed out before, the number name 'five must always stand between four and 'six" If 'five' were variable in its position and could stand sometimes next to seven' and at other times next to 'eight," the series would lose its character as a fixed series. This essential property of the number series is not the first fact which children of modern times learn They are introduced to the number names which society has perfected long before they have any of the needs which led to the use of number names Furthermore children are not required to pass through the stage of learning concrete number

names, because language has long since passed into the abstract stage. Number names are used by children at first in purely accidental order. They count by saying, "four, seven, three, six," wholly unaware of the serial arrangement or fixity of order of the individual number names In this respect, children do not follow the order of racial evolution. They are introduced to an elaborate ready-made series of names, and it requires time and experience to acquaint them with the significance of these names

WRITTEN SYMBOLS

The racial evolution of the number series was greatly aided, if not indeed made possible, by an invention which paralleled the invention of number names This second invention of major importance was the series of written symbols which express the various numbers. Early civilizations exhibit interesting experiments in the evolution of methods of recording numbers For purposes of the present discussion, it is not necessary to go back of the system of symbols used by the Romans The first three numbers, it will be recalled, are represented in that system by single straight lines Evidently this simple device reaches its limit of usefulness when four tallies are passed Indeed, if more than three lines are drawn, the eye begins to be confused, the picture must therefore be simplified A type of simplification which is adopted in our own day when one is using straight lines as tallies is that of binding together four tallies by a fifth, which runs obliquely across the four This may explain the origin of the Roman V. In any case, it helps us to understand why the Romans adopted a new symbol for five after using straight lines as tallies for the lower numbers

EARLY SYSTEMS ENCUMBERED BY VARIETY OF SYMBOLS

The characteristic fact in all primitive number systems, including the Roman, is that there are special symbols for all the higher units There is X for ten, L for fifty, C for one hundred, D for five hundred, and M for one thousand The use of special symbols for the higher numbers corresponds in part to the fact that in the case of the series of number names it became necessary to invent wholly new names, such as "hundred" and "thousand," for the higher denominations The interesting point is that the number symbols used by the Romans are even more manifold in form than is the series of number names Fifty, which in its name shows a relation to five, is recorded by a symbol wholly different from the symbol which represents five In like fashion, the symbol for 500 obscures entirely the relation of this number to 5 or 50

The Roman system is clumsy in spite of the fact that it is an advance beyond all the systems which preceded it How utterly impossible the numerical expressions of modern business and science would be if we were compelled to depend on the Roman system of numerals will be readily recognized if one tries to imagine stock quotations couched in Roman numerals or an example in multiplication in which DLV is the multiplicand and XIX is the multiplier

THE ARABIC SYSTEM A HIGHLY PERFECTED SYSTEM

What was necessary in order that a system of symbols should serve the requirements of number thinking to the fullest possible extent was the evolution of a system of symbols which is easy to use in making combinations The Arabic numerals have the virtue of representing all that is represented in the series of number names and at the same time suggesting the strictly cyclic character of the number series By its very form 50 is shown to be subject to all the rules of combination which belong to 5 The Roman L confuses thinking The Arabic 50 clarifies thinking

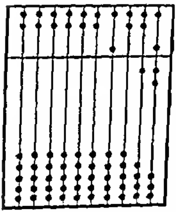


FIG 1 —An abacus

The origin of the Arabic numerals is lost in the obscurities of Oriental history Somewhere in India the genius of man produced this system, which is one of the most important intellectual possessions of the race It is suggested that the invention of the Arabic numeral series was the outcome of experiments in social organization The Indian villages are organized in such a way that there is a head man for every group of ten families This head man is in turn the lesser unit in an organization which controls ten communities A head man is a unit of higher order than the single family but a unit of lower order in his relation to the larger organization to which he is subordinated

Another suggestion and probably a more plausible one is that the Arabic system grew out of experimentation with the abacus The abacus is a calculating device used even today in many parts of the East in adding numbers The abacus operates on the principle that one row of tallies is to be used in counting up to ten and a second row of tallies is to be used to record the tens up to one hundred a third row of tallies is to be used

to record hundreds, and so on Figure 1 shows a common form of the abacus The string of beads at the right records units the second string records tens and so on The upper and lower divisions are employed to simplify the use of numbers less than ten, the beads in the upper division representing in each case five beads in the lower division The beads are set in the figure to represent 5 017 The idea that position is important in determining the value of a counter is clearly brought out by this contrivance Each bead is like every other bead when it is considered without regard to its position Each row of beads is, on the other hand wholly different from every other row The ultimate significance of a unit bead is derived from its relation to the whole collection of units

SPECIAL SIGNIFICANCE OF ZERO

Especially interesting in this connection is the fact that the Arabic number system would be impossible without the symbol o which holds the tallies in place much as the wires of the abacus hold the beads of different denominations in place Zero is as teachers know very well, a source of great confusion in the minds of pupils learning how to make combinations which include a mere place holder always causes pupils trouble

NUMBER AS AN ABSTRACT SYSTEM

In general, we find that pupils experience great difficulty in the mastery of the complex number system which modern civilization provides for them The mature number system has lost all its primitive characteristics It is no longer concrete it does not use analogies with familiar objects it does not even use the simple tallies with which the Romans began their system The Arabic numerals are very remote from primitive experience Pupils are introduced at an early age to this highly evolved product of generations of intellectual effort with the expectation that they will somehow adopt the perfected system and learn to employ it Perhaps the most impressive way of stating the situation is to point out that the number system is a highly abstract system It has no direct connections with the child s experience with objects It is a system that has evolved to the point where all its references to objects have been lost and only the sheerest abstractions remain

THREE WAYS OF EXPRESSING NUMBERS

The number system offers one difficulty which is unique in the world of intellectual difficulties There are two wholly different ways of making a written record of numbers One method is that of spelling out number

names the other is that of using symbols which are not names but simplified substitutes for names Thus the child must learn to pronounce the word 'six', he must learn to recognize the written word 'six' and he must also learn to recognize and to use fluently the symbol 6 In this respect, arithmetic is much more complicated than ordinary reading There are two steps in reading The pupil must understand and pronounce each word He must be able to recognize the written word and to relate it to the corresponding spoken word Here the substitution of verbal symbols for objects ends In arithmetic, as contrasted with reading there is a third step away from the object In arithmetic there is a numeral to be recognized and interpreted

SIGNS OF COMBINATION

In the course of our later investigations we shall emphasize the fact that arithmetic goes beyond the number symbols The plus sign is added to the number symbols The plus sign is not a number, and it is not a synonym for a single word The plus sign is represented in verbal problems by many different words such as 'add' 'take together' and 'combine' The plus sign is the symbol of a relation which is to be set up among symbols Other similar additions to the symbolism of arithmetic are found in the signs of substraction multiplication, and division

NUMBER NOT INSTINCTIVE

It is certainly not a matter to be wondered at that pupils have difficulty with arithmetic The race has put into the science of number its mature experience gathered and refined through long ages of experimentation The child comes into the world utterly destitute of any native desire or form of expression corresponding to the Arabic number system There is no instinct of number there is no native tendency to pass from number names to abstract symbols The child must be led by such instruction as the school can provide to a comprehension of this elaborate intellectual system, which the race has worked out History shows that the modern child is offered a number system superior to anything which European nations were able to invent The child is given the benefit of a fortunate borrowing by Europe of the most advanced thinking of the Orient The schools find that human nature often resists the process of introduction to the complications of arithmetic For generations teachers have been baffled by the problem of giving to pupils in a short time the results of generations of human thinking There have been many failures in the teaching of number These failures attest the difficulty of

that which education attempts In spite of difficulties and past failures, however the school is charged with the duty of renewing its efforts to accomplish its task If pupils do not come into the world with number as a native endowment, they must cultivate ideas of number by the exercise of that power of superseding inheritance which has been given to human beings namely, the power of learning methods of abstract thought If teachers find the teaching of arithmetic difficult, they must analyze their problem and come to a fuller understanding of the mental processes of pupils in order that they may more effectively guide the pupils through the complexities of the number system

PURPOSE OF THE PRESENT STUDY

This monograph aims to contribute in a measure to the analysis of number ideas It will describe a series of laboratory experiments and analytical studies which were undertaken with a view to discovering the mental processes through the development of which pupils arrive at an understanding and correct use of some of the more fundamental arithmetical ideas

The investigations here reported do not supply solutions of all the educational problems which confront teachers of arithmetic It is believed, however, that they are sufficiently promising in technique and in findings to justify their publication as a step in a direction which must be followed in securing a basis for new methods of teaching number to pupils

CHAPTER II

COUNTING BY ADULTS

STANDARD TESTS

The method of investigating the achievements of pupils in arithmetic which has been most commonly employed during the last twenty five years is the method of testing individuals and groups by standardized tests and of comparing the scores made on these tests Through the use of tests, much new knowledge has been gained about the way in which pupils deal with number For example, it has been clearly demonstrated that ability in one arithmetical operation, such as addition, may be well developed in an individual who is much below the average in some other phase of arithmetic, such as subtraction Furthermore, it has been shown that there are wide differences between individuals in the same school grade Many pupils in the upper grades show only such degrees of mastery of arithmetical operations as correspond to the averages for pupils in the lower grades

ANALYSIS OF TEXTBOOKS, COMPARISON OF JUDGMENTS, AND COMPLETE ENUMERATION OF CASES

The findings of the tests have raised grave questions regarding the efficiency of present methods of teaching number In the effort to improve the teaching curriculum studies have been added to the investigations made by the testing method A number of inquiries have been made into the nature of the exercises and problems prescribed by the textbooks which are in most common use The opinions of experienced educators have been ascertained with regard to the retention or elimination of various topics in the conventional course Lately, a number of exhaustive studies have been made of all the possible number combinations in addition subtraction, multiplication, and division It is held by some of those who carry on this type of investigation that each particular combination is a separate item of experience and must be taught as such in order to give the pupil complete training in the fundamental operations.

PRACTICAL ARITHMETIC

The curriculum studies have also been extended to cover the practices of society outside the schools The question has been raised What arithmetic does the ordinary man or woman need in everyday life? The strik-

ing fact which issues from these studies of the practical uses of arithmetic is that very little arithmetical knowledge is really needed by most people The growing tendency which seems to issue from such studies is advocacy of a drastic reduction of attention to arithmetic in the schools

NEGLECT OF PSYCHOLOGICAL ANALYSIS OF ARITHMETICAL PROCESSES

During the period of testing and of analyzing the curriculum, little attention seems to have been given to a study of the mental processes through which pupils pass when they try to acquire a knowledge of number Some of the earlier studies of arithmetic discussed such questions as the origin of number ideas and the form of thinking adopted by pupils in solving number problems One of the debates which was carried on vigorously during the nineties dealt with the question whether number ideas originated in measurement or in counting The evidences offered pro and con were very general in character as were all the educational discussions of that day

Very little has been done during the last thirty five years by way of direct investigation of the mental processes involved in the development of number ideas and in the application of these ideas to practical situations The reason for this dearth of psychological analyses is to be found in part in the preoccupation of students of education in tests and in curriculum studies and in part in the difficulty of devising methods by which psychological analyses can be made

NUMBER CONSCIOUSNESS AND VISUAL EXPERIENCE WITH GROUPS OF OBJECTS

One type of analysis of number experiences which has been attempted by a few investigators and which has in general been found to be relatively unproductive is that which employs a group of visual objects as the material presented to the pupils Series of dots on paper have been used The dots have been arranged in one form or another and have been exposed to view for short intervals of time and pupils have been asked to judge how many were exposed This method has not been effective because the space factors in the situation have always proved to be of preponderant importance in determining the results When the dots are arranged three in a group at the apexes of an imaginary equilateral triangle, or when they are arranged four in a group at the corners of a square or when they are arranged five in a group as in the ordinary domino arrangement of five spots recognition of the number is dependent not so much on counting as on the recognition of the form

ARITHMETICAL PROCESSES SUBJECTIVE

The difficulty arising from the confusion of number experiences with other phases of consciousness which is illustrated in the foregoing paragraph is greatly augmented by the difficulty of securing from pupils any account of the way in which they think about numbers. The characteristic fact about number consciousness is that it is conditioned very little by external impressions or by reactions which can be readily recorded. Number processes, like all the higher mental processes, are peculiarly internal processes open to inspection only to the individual who is having the conscious experience.

It may not be out of place to reiterate in this connection what was said in the last chapter about the difficulty encountered in teaching arithmetic because the teacher does not know what is going on in the pupils' minds. The complexities of the processes involved in reading numerals which were revealed by the investigations of Terry[1] and the great variety of complicated mistakes reported by Buswell[2] as occurring when pupils attempt to deal with the four fundamental operations are largely overlooked by teachers because *number consciousness is subjective to a more extreme degree than are any of the other forms of experience with which elementary education deals.*

METHODS OF THE PRESENT INVESTIGATION

The foregoing references to earlier investigations of the number experiences of children prepare the way for an explanation of the methods which were adopted in the present inquiry. These methods were devised with a view to securing numerous easily recorded reactions from the individuals under investigation. They were arranged so as to provide a wide range of number experiences which could be readily controlled by the experimenter. Because of the unsatisfactory results secured with visual objects arranged in spatial patterns, the methods adopted avoided altogether the combination of number with space.

Briefly described, the methods adopted were as follows. The subject or person on whom the experiment was tried was seated in a quiet room. He was asked to give attention either to a series of sounds or to a series

[1] Paul Washington Terry, *How Numerals Are Read: An Experimental Study of the Reading of Isolated Numerals and Numerals in Arithmetic Problems.* Supplementary Educational Monographs, No. 18. Chicago: Department of Education, University of Chicago, 1922.

[2] G. T. Buswell, with the co-operation of Lenore John, *Diagnostic Studies in Arithmetic.* Supplementary Educational Monographs, No. 30. Chicago: Department of Education, University of Chicago, 1926.

of flashes of light The sounds or flashes of light were controlled in rate by apparatus in an adjoining room The sounds or flashes of light were varied in rate and number They were sometimes given at the rate of two per second, sometimes they were given more rapidly at the rate of three, four, five, or more per second The intervals between sounds or flashes of light in any given series were always uniform For example, the interval between sounds when they came at the rate of three per second was always one third of a second The number of sounds or flashes of light given in successive trials was varied, sometimes three sounds or flashes of light were given sometimes as many as twenty In each case the subject, or person under investigation, was asked to record the number of sounds or flashes of light that he observed The usual procedure was to give at first a series of sounds or flashes of light at a slow rate, such as one every third of a second and then to increase the rate step by step until the rate was one every seventh or eighth of a second at which point the limit of the subject's ability to count correctly was usually passed

APPARATUS

The sounds used in the experiments were produced by an electric sound hammer This is a familiar piece of apparatus used in the psychological laboratory It consists of an electromagnet which causes a small steel hammer to strike against a metal base whenever an electric current passes through the magnet By means of an electric current under his control, the experimenter can thus produce sharp clear sounds at any desired rate and of any desired number The flashes of light were produced by means of an electric current passing through a Geissler tube The apparatus necessary to control the sound hammer or the Geissler tube is somewhat complicated and the general reader who is not interested in laboratory technique should perhaps be advised to omit the description which follows

The control apparatus is placed in a room adjoining that in which the subject sits This control apparatus consists of a Dunlap ten pole synchronous motor which is driven by an electric current interrupted by an electric tuning fork of the frequency of fifty vibrations per second The motor supplied by the C. H Stoelting Company, was remounted with ball bearings so as to secure the maximum of energy for driving the rest of the apparatus

On the axis of the motor is mounted a large aluminum disk, eighteen centimeters in diameter In front of this disk and perpendicular to it is mounted a fiber wheel ten centimeters in diameter, which can be adjusted

along a shaft When brought against the aluminum disk, the fiber wheel is driven by friction In order to insure firm contact between the disk and the wheel, the latter is held against the aluminum disk by a spring The fiber wheel can be adjusted along its shaft so as to be in contact with the disk near its center, when the wheel rotates very slowly, or the wheel can be brought into contact with the disk near its periphery, when the wheel rotates rapidly The range of adjustment is ample to provide sounds or flashes of light of the type required by the experiment from two per second to twelve per second

When it is adjusted in any desired position, the friction wheel can be made fast to its shaft by a set screw Mounted on the end of this shaft are two hard rubber wheels into which are set metal strips When the apparatus is in rotation, these strips pass under metal brushes and thus make and break electric circuits The rate at which circuits are made and broken depends on the rate of motion of the shaft, and this depends in turn on the position of the fiber wheel

Two independent electric circuits are controlled by the two hard rubber wheels, these circuits being made whenever the metal strips pass under the brushes and being broken as soon as this contact is interrupted One of these circuits passes from the control apparatus to the electric hammer described in an earlier paragraph, the other passes through an induction coil to the Geissler tube On the way to the stimulus producing apparatus, the current passes through a marker, which makes a record each time the circuit is made and broken The current also passes through a key, which is used by the experimenter to turn the current off entirely or to send it through the sound hammer or the induction coil

The marker is provided with a fountain pen The rubber tube which holds the ink and the section of the case carrying the pen are the only parts of the fountain pen used The fountain pen is light enough to be readily moved by the marker An ink line record is traced on a strip of paper, which is unrolled over a rotating drum at the rate of one inch per second This recording apparatus makes it possible for the experimenter to secure an exact count of each of the series of sounds or flashes of light which he gives to a subject

The current used in producing flashes of light is somewhat more difficult to control than that which operates the sound hammer The light-producing current is conducted from the make and break apparatus to the primary coil of an induction coil The secondary coil is connected with the Geissler tube, which is placed before the subject. Rotation of the motor thus produces, whenever the experimenter desires, flashes of light

controlled with regard to their rate and number just as are the sounds which the sound hammer produces The number of flashes of light given in any case is recorded by a marker as described in the case of the sounds

In addition to the apparatus thus far described, there is a buzzer in the room with the subject which is controlled by a key in the room with the experimenter By means of this buzzer the experimenter can warn the subject in the next room that a series of sounds or flashes of light is about to begin

PROCEDURE IN THE EXPERIMENT

The procedure in experimentation is as follows The experimenter sets the Dunlap motor in rotation and adjusts the friction wheel in such a position that it can make the sound hammer or the Geissler tube operate at a rate chosen as the rate at which the experiment is to begin as for example, a rate of three times per second He also starts the drum on which a record is to be taken of each time the hammer is made to sound or a flash of light is made to appear When all is in readiness he gives the subject a warning signal with the buzzer and closes the circuit between the hard rubber wheel which acts as an interrupter and the sound hammer or the Geissler tube He allows the circuit to remain closed until the subject has been given four five or more sounds or flashes of light He then opens the circuit and allows an interval to elapse while the subject records the number of sounds or flashes of light which he observed After this interval which is usually from fifteen to twenty seconds the experimenter once more sounds the buzzer closes the circuit and gives the subject another series of sounds or flashes of light After several series of different lengths have been given at the rate of three per second the friction wheel is readjusted and brought into a position where it can make the electric circuit four times per second The subject is then given several series at this more rapid rate Further adjustments give the subject series at the rate of five per second six per second and so on

It is also possible to give series of sounds and flashes of light simultaneously When sounds and flashes of light are given simultaneously the rates of the two series are identical

ERRORS MADE BY FIVE SELECTED SUBJECTS IN COUNTING VARIOUS SERIES OF SOUNDS

Table I presents the results secured in the case of five subjects when sounds were given at different rates and in series of different lengths The five subjects were selected from a group of forty graduate students of education who participated in the experiment The subjects whose records

TABLE I

ERRORS MADE BY FIVE SUBJECTS IN COUNTING SERIES OF SOUNDS

NUMBER OF SOUNDS GIVEN	ERRONEOUS NUMBERS REPORTED BY FIVE SUBJECTS					NUMBER OF ERRORS
	Subject 15	Subject 13	Subject 6	Subject 36	Subject 30	
Three sounds per second						
5				6	4	2
7					8	1
11						0
4						0
14			3		13	1
7						0
9						0
6					4	1
12					10	1
Number of errors	0	0	0	1	5	
Four sounds per second						
6					5	1
5					4	1
14					12	1
4					3	1
10				11	8	2
6				7	5	2
17		16	16	16	15	4
7				8	6	2
11			10		10	2
Number of errors	0	1	2	4	9	
Five sounds per second						
6					5	1
13				8	11	2
5					4	1
10			9	8	9	3
12			11	9	11	3
7				6	5	2
11				8	10	2
9				7	7	2
11			10	9	10	3
Number of errors	0	0	3	7	9	
Six sounds per second						
6					7	1
5						0
8			7	7	7	3
13			12	9	10	3
11			10	10	10	3
8			7		7	2
14			13	10	11	3
9			8		7	2
14			13	11	11	3
Number of errors	0	0	7	5	8	

TABLE I—*Continued*

Number of Sounds Given	Erroneous Numbers Reported by Five Subjects					Number of Errors
	Subject 15	Subject 23	Subject 6	Subject 36	Subject 30	
Seven sounds per second						
[illegible]				6	5	2
7		6			5	2
6					3	1
11			10	10	10	3
8				5	6	2
12		13	11	10	9	4
11			10	10	10	3
Number of errors	0	2	3	5	7	
Eight sounds per second						
7				6		1
6				5	7	2
13		12	11	10	10	4
6				5	5	2
11		10	10	10	10	4
7				6	5	2
16		14	12	*	10	4
Number of errors	0	3	3	7	6	
Total number of errors	0	6	18	29	44	

* No number reported

are reported in the table were selected to show marked differences in accuracy and are arranged in the order of their efficiency in counting

FAILURE TO MATCH SUBJECTIVE AND OBJECTIVE SERIES

The most striking fact which appears from a study of Table I is the fact that mature persons differ greatly in their ability to count A second fact is that, when errors in counting appear they are most commonly of a type which indicates that the subject lagged behind the objective series In some cases the subject was quite certain that he was right when in reality he was one or more counts deficient Such cases throw light on the process by which the deficiency was caused Evidently the subject had to divide his attention between two series the subjective series of numbers which he employed in counting and the objective series of sounds to which he tried to match his subjective series In trying to pay attention to the two series he overlooked in some cases some item or items in the objective series He was often not aware of the lack of agreement between his subjective counting and the objective sounds

OVERCOUNTING

There were some cases in which the observer was aware of the fact that the two series were not coinciding and was led by this awareness to continue the subjective counting after the objective series had stopped It will be noted from Table I that confusion of this kind more commonly appears in the case of relatively small numbers The only number above 10 which is shown by the table to be overcounted is the 12 in the series given at the rate of seven per second, Subject 13 reported 13 in this case

TABLE II

TOTAL NUMBER OF ERRORS MADE BY EACH OF FORTY SUBJECTS IN COUNTING THE SERIES OF SOUNDS LISTED IN TABLE I

Subject	Number of Errors	Subject	Number of Errors
1	2	21	35
2	11	22	9
3	3	23	7
4	15	24	44
5	36	25	23
6	18	26	26
7	1	2	8
8	15	28	10
9	3	29	29
10	12	30	44
11	20	31	12
12	22	32	14
13	6	33	22
14	2	34	9
15	0	35	21
16	8	36	9
17	13	37	[illegible]
18	14	38	17
19	6	39	8
20	8	40	13

SPECIAL DIFFICULTY WITH LARGE NUMBERS

Inspection of Table I brings out clearly the fact that miscounting occurs most frequently in the case of the larger numbers Several explanatory considerations suggest themselves The opportunities for error are, of course more numerous in a long series than in a short series The tax on attention is relatively greater in a long series Added to these explanations is one which was discovered in the experiments described in the last chapter and which is confirmed by experiments to be reported in subsequent sections of this monograph The subjective series of number names becomes very difficult to repeat beyond ten that is when the subject begins to use number names which have more than one syllable

CONFIRMATION BY REFERENCE TO RECORDS OF FORTY SUBJECTS

Tables II and III summarize the results for the whole group of forty subjects from which the five whose records are reported in Table I were selected All the statements which were made in discussing Table I are here confirmed

TABLE III

TOTAL NUMBER OF ERRORS MADE BY FORTY SUBJECTS IN COUNTING SERIES OF SOUNDS

Number of Sounds Given	Number of Errors	Number of Sounds Given	Number of Errors
Three sounds per second		Six sounds per second	
5	3	6	7
7	1	5	8
11	2	8	12
4	1	13	25
14	4	11	18
7	0	8	12
9	0	14	30
6	2	9	18
12	3	14	22
Four sounds per second		Seven sounds per second	
6	4	7	17
5	3	7	14
14	8	6	6
4	2	11	23
10	9	8	19
6	5	12	27
17	17	11	22
7	7	Eight sounds per second	
11	8	7	17
Five sounds per second		6	14
6	4	13	35
13	15	6	15
5	5	11	31
10	12	7	18
12	12	16	38
7	9		
11	14		
9	11		
11	13		

INFERIOR ABILITY EXHIBITED IN COUNTING FLASHES OF LIGHT

After the tests with series of sounds were completed the same subjects were tested with regard to their ability to count successive flashes of light Tables IV V and VI present the results in such form that a direct comparison can be made with Tables I II and III respectively Counting flashes of light is evidently a much less accurate performance than counting sounds This result is amply confirmed by all subsequent

results secured when flashes of light were used A number of experiments with flashes of light will be reported in later sections of this monograph

TABLE IV

ERRORS MADE BY FIVE SUBJECTS IN COUNTING SERIES OF FLASHES OF LIGHT

NUMBER OF FLASHES OF *LIGHT GIVEN*	ERRONEOUS NUMBERS REPORTED BY FIVE SUBJECTS					NUMBER OF ERRORS
	Subject 15	Subject 13	Subject 6	Subject 36	Subject 30	
Three flashes per second						
4					3	1
6			5		5	2
12					10	1
9					8	1
7			6		6	2
10				9	9	2
Number of errors	0	0	2	1	6	
Four flashes per second						
6		5	5		5*	3
5			4	4		2
12		11	10	9	13	4
4			3			1
12	11	11	11	10	11	5
7	6	5	4	6	6	5
9	6	6	5	7	6	5
Number of errors	3	5	7	5	5	
Five flashes per second						
4	3	3	3	3	3	5
6	5	5	4	5	4	5
12	8	7	7	*	10	5
6		4	4	5	5	4
9	8	7	7	8		4
12	9	9	10	*	9	5
6	5	4	5	*	4	5
Number of errors	6	7	7	7	6	
Total number of errors	9	12	16	13	17	

*No number reported

SENSORY CONDITIONS NOT EXPLANATORY OF RESULTS WITH FLASHES OF LIGHT

The difficulty in counting flashes of light is in no measure attributable to the character of the visual sensations involved In order to guard against the possibility that after images might complicate the counting a number of variations were introduced in the case of three subjects

Their counting of flashes of light was tested (1) in a well lighted room with the flashes of light seen through a small aperture, (2) in a well lighted room

TABLE V

TOTAL NUMBER OF ERRORS MADE BY EACH OF FORTY SUBJECTS IN COUNTING THE SERIES OF FLASHES OF LIGHT LISTED IN TABLE IV

Subject	Number of Errors	Subject	Number of Errors
1	9	21	9
2	4	22	10
3	7	23	10
4	11	24	10
5	15	25	15
6	16	26	9
7	11	27	11
8	13	28	12
9	14	29	14
10	9	30	17
11	8	31	10
12	11	32	9
13	12	33	10
14	9	34	13
15	9	35	8
16	14	36	13
17	9	37	9
18	11	38	14
19	9	39	18
20	10	40	17

TABLE VI

TOTAL NUMBER OF ERRORS MADE BY FORTY SUBJECTS IN COUNTING SERIES OF FLASHES OF LIGHT

Number of Flashes of Light Given	Number of Errors	Number of Flashes of Light Given	Number of Errors
Three flashes per second		Four flashes per second—Continued	
4	5	7	37
6	4	9	40
12	8	Five flashes per second	
9	7	4	37
7	5	6	38
10	10	12	40
Four flashes per second		6	35
6	18	9	32
5	13	12	33
12	28	6	36
4	9		
12	27		

with the flashes of light seen through a large aperture and (3) in a dark room with the flashes of light seen through a large aperture. The purpose

of using a larger aperture was to produce a stronger stimulation. The purpose of darkening the room was to make the flashes of light relatively much brighter and thus to increase even more than did the enlargement of the aperture the tendency to produce after images. If after images were in any degree the cause of the difficulty in counting flashes of light, the first series of tests should have produced the least number of errors and

TABLE VII

ERRORS MADE BY THREE SUBJECTS IN COUNTING SERIES OF FLASHES OF LIGHT GIVEN UNDER VARIOUS CONDITIONS AT THE RATE OF FOUR PER SECOND

Number of Flashes of Light Given	Erroneous Numbers Reported by Three Subjects			Number of Errors
	Subject F	Subject G	Subject H	
Flashes in lighted room seen through small aperture				
5			4	1
8	7	6	7	3
7	6	6	5	3
6	5	4		2
Number of errors	3	3	3	
Flashes in lighted room seen through large aperture				
5				0
8	7	7	6	3
7	6	6	6	3
6	5	5		2
Number of errors	3	3	2	
Flashes in dark room seen through large aperture				
5				0
8		7	7	2
7	6	6		2
6			5	1
Number of errors	1	2	2	

the second and third series of tests should have produced successively larger numbers of errors. *Exactly the opposite was found to be the case.* Table VII shows the errors made by the three observers. That part of this table which shows the errors made when the flashes of light were seen in a lighted room through a large aperture corresponds well with the section of Table IV which shows the results secured with five subjects when they were counting flashes of light given at the rate of four per second.

LOW LEVEL OF EFFICIENCY IN COUNTING FLASHES OF LIGHT ATTRIBUTABLE TO DEFICIENCY IN EXPERIENCE

If the difficulty in counting successive flashes of light is not attributable to the character of the sensory experience, it must be attributed to the inability of the subject to divide his attention readily between the subjective series of number names which he uses in counting and the flashes of light The greater difficulty experienced by all subjects in counting flashes of light as contrasted with the difficulty experienced in counting sounds must be related in some way to the development of experience The ability to repeat the subjective number series must be thought of as fairly constant for any given individual We are driven to the conclusion that the application of the subjective series to the counting of sounds is a more highly trained function than is the application of this same subjective series to the counting of flashes of light It may be pointed out that throughout life the number series is intimately associated with sounds The child learns number names by repeating them orally, he thus hears as well as utters the number names The amount of experience which the ordinary person has in counting flashes of light is relatively small Consequently, visual series are found to be much more difficult to count

EXTREMELY LOW LEVEL OF EFFICIENCY IN COUNTING TACTUAL IMPRESSIONS

Some experiments were tried in which a series of taps on the back of the hand were substituted for the series of sounds or flashes of light There were a number of complications which tended to make this series of experiments somewhat unsatisfactory First it was very difficult to adjust the intensity of the taps so as to avoid fatigue of the skin at the point tapped Second it was impossible to avoid accompanying sounds altogether The sounds were obviated by stopping the subject s ears This seemed to be seriously distracting to one of the subjects Finally, in order to secure correct counting, it was necessary to use a rate slower than any rate employed in the series of sounds or flashes of light It was necessary also to use very short series of taps

The results of a series of tests of four subjects when they were given taps on the back of the hand are reported in Table VIII This table should be compared with Tables I and IV It will be seen that the subjects were far less competent in counting taps than in counting flashes of light Evidently, counting skin sensations is a relatively less familiar operation than counting flashes of light and therefore a far less familiar operation than counting sounds

COUNTING ONE'S OWN TAPPING MOVEMENTS

Experiments were tried substituting for passive reception of tactual stimulations on the back of the hand a series of active taps made by the subject. The subject in this case received tactual stimulations from the

TABLE VIII

ERRORS MADE BY FOUR SUBJECTS IN COUNTING SERIES OF TAPS ON THE BACK OF THE HAND

Number of Taps Given	Erroneous Numbers Reported by Four Subjects				Number of Errors
	Subject F	Subject H	Subject K	Subject L	
Two taps per second					
8	6	7	7	6	4
11	10	9	10	8	4
4		3			1
2					0
5	4		6	4	3
3				4	1
6	5	4	*	4	4
7	6	5	*	5	4
3		4			1
Number of errors	5	6	5	6	
Three taps per second					
3		4			1
3			2		1
4	5	3	3	6	4
4		3	5		2
6	4	5	7	4	4
8	6	6	9	5	4
5	4	4	4	3	4
9	7	6	7	7	4
Number of errors	5	7	7	5	

*No number reported

tapping finger, kinaesthetic sensations from the muscles and joints involved and such experiences as came from the sending out of the motor impulses which produced the tapping movement. Three subjects were tested. *Table IX* presents the results. The subjects were required to make series of tapping movements as rapidly as they could and to report in each case the number of such movements.

It was found that all the subjects tested could count their own tapping movements with a high degree of accuracy even when the rate of these movements was rapid. The only series which were inaccurately counted

is relatively ineffective A few tests made with very long series of tapping movements in which the subject was asked to count by using the number names up to 'ten' only, repeating these simple names as many times as necessary, indicated that a person can count his own tapping movements with accuracy to any number and at any rate that he can produce so long as he uses simple number names

GENERAL CONCLUSIONS

The experiments thus far reported justify the formulation of several general statements First, there are great individual differences in the ability to count Second a given individual exhibits wide differences in ability in counting different types of experiences Third the competency of any individual in counting series of experiences more than ten in number is much less than his competency in counting short series The difficulties encountered in counting long series do not seem to arise altogether from the tax on attention because long series can be counted accurately if number names less than ten are employed Apparently the difficulty in counting long series is due in very large measure to the complicated number names which are used in counting beyond ten The foregoing statement seems to justify a fourth conclusion namely that counting is a process consisting in a succession of inner acts which are in essence inner articulations of number names

RATE OF REPEATING THE SERIES OF NUMBER NAMES

The importance of the fourth conclusion stated in the preceding paragraph is so great that a series of tests was undertaken with five subjects for the purpose of testing its validity This series of experiments consisted in securing a measurement of the rate at which each subject could repeat various number series orally and silently The tests here under discussion differ from the earlier tests in that the person under examination was not asked in this case to count anything but merely to repeat the number series

The method of this series of experiments was simple The subject was seated before a table on which was fastened a reaction key He pressed down the key and began to count When he had completed the series he lifted his finger from the key In order to reduce to a minimum the error due to the fact that the hand reaction itself requires some time the subject was required to make a series of such reactions counting continuously and making a reaction with the hand at the end of each series Thus

if the subject was counting up to "ten," he proceeded as follows: He pressed down the reaction key and began to count "one, two, three," and so on up to "ten," holding the key down during the counting. On reaching "ten," he lifted his finger but went on counting, beginning once more at "one." On reaching "ten" the second time, he once more pressed down the key and continued to count, beginning with "one." He repeated the series until he had made enough reactions to eliminate the error introduced by the hand reaction. In a similar way, the time was determined for series of counts running from "one" to "twenty," from "one" to "thirty," from "eleven" to "twenty," and from "twenty-one" to "thirty." Table X presents the facts secured in a number of such experiments. The time records

TABLE X

Average Number of Seconds Required by Five Subjects in Counting Various Series

Method of Counting and Number Counted	Subject M	Subject N	Subject O	Subject P	Subject Q
Counting orally from "one" to "ten"	1 2	1 4	1 1	1 1	1 6
Counting orally from "one" to "twenty"	3 4	3 7	3 0	3 2	4 1
Counting orally from "eleven" to "twenty"	2 4	2 9	1 9	2 0	2 7
Counting orally from "twenty-one" to "thirty"	1 9	1 5	1 3	1 6	2 2
Counting orally from "one" to "thirty"	5 4	4 8	4 2	5 1	5 7
Counting silently from "one" to "ten"	1 3	1 2	0 9	1 4	1 3
Counting silently from "one" to "twenty"	3 0	3 2	2 4	3 1	3 6
Counting silently from "eleven" to "twenty"	1 8	1 9	1 6	1 9	2 4
Counting silently from "twenty-one" to "thirty"	1 6	1 2	1 3	1.7	2 1
Counting silently from "one" to "thirty"	4 2	4 2	3 3	5 0	5 3

were supplied by a Jacquet graphic chronometer The record could be read accurately in tenths of a second Each time record is an average of the records in from three to five repetitions of the series counted.

THE SUBJECTIVE COUNTING SERIES

Table X reveals a number of facts which are very important for our understanding of the counting process First, a comparison shows that, in general, there is a very close relation between the rates of oral and silent counting in the case of all the subjects Subjects O and Q counted somewhat more rapidly silently, while Subject P was in some cases slower in silent counting In the main, however, the rates in silent and oral counting show that the processes are much the same Introspection confirms this conclusion, almost every person finds that silent counting involves incipient articulation of the number names

The second fact which is exhibited clearly and without exception by Table X is that counting from "eleven" to "twenty" is always slower than counting from "one" to "ten" This statement holds for silent counting as well as for counting aloud This fact can have only one explanation Counting from "eleven" to "twenty" is an elaborate process, because the number names from "eleven" to "twenty" have more than one syllable and, under all conditions, require for their use more expenditure of nervous energy than is required for the use of the shorter number names from "one" to "ten"

Third, the numbers from "twenty one" to "thirty" were evidently dealt with by the various subjects in different ways Subject P, who was rapid in counting was slow in counting from "twenty one" to "thirty" His introspections showed that he tended to articulate the word "twenty" quite fully Subject N was fast in counting from "twenty-one" to "thirty" He accounted for this fact by saying that he neglected the word "twenty" This habit was exhibited in his oral counting, for he counted by saying "twenty one two three, four," etc

Table X shows clearly that counting is an active process involving, in some degree, the articulation of the number names We may speak of a person's ability to count as the possession of a subjective series In the course of education, one learns to make a series of articulations in regular order The fluency with which one can repeat this subjective series varies in different individuals but the subjective series is much the same whether the person says the words aloud or thinks them through in his mind In other words, counting is a process depending on the possession of a certain motor habit

APPLICATION OF THE SUBJECTIVE SERIES TO OBJECTIVE SERIES OF VARIOUS KINDS

The findings reported in this chapter can be summarized briefly as follows The tests with sounds, flashes of light and tactual experiences show that the subjective counting series cannot be applied at its maximum rate to lights and tactual experiences and that it cannot be applied to sounds at its maximum rate by most persons, although in the counting of sounds the rate approaches and sometimes equals the rate of the person s ability to repeat the number names A very close relation is found between the two motor series, that of tapping movements and that of counting The relation between counting and tapping is even closer than the relation between counting sounds and the subjective counting series The rates of oral and silent counting are much alike, and the rate of

counting with long number names is always slower than the rate of counting with short number names

EXPERIENCE AS A SERIES OF ORGANIZED PATTERNS

The facts reported in this chapter have a significance for the science of education much broader than that which appears when they are thought of merely as facts about counting It has been shown that different types of counting have what may be called different 'psychological patterns' Each type of counting is a complex experience depending for its special character on the individual s training and on the method of procedure involved The psychological understanding of counting requires a study of the various organized patterns exhibited in each act of counting

As it is with counting so is it with every mental process that is cultivated in the school or in individual learning Education is a maturing of complex organized patterns of consciousness Education cannot become truly scientific until it discovers by the most searching inquiry the different patterns of experience The popular use of such a word as ' counting to cover a number of different kinds of mental activity tends to be adopted by teachers Too often they think of all counting as one type of intellectual activity As has been amply demonstrated in this chapter investigation shows that the various forms of experience which are usually classified together under such a term as counting are in fact highly complex and clearly distinguishable patterns of mental and physical behavior

The meaning of such findings for educational practice is not obscure Teachers must learn to carry their studies of pupils far enough to gain an insight into the differences which appear in mental patterns Teachers must not be misled by ordinary thinking which indiscriminately classifies together forms of mental activity that are essentially unlike Through a type of scientific insight which is much more fully developed than is ordinary popular thinking teachers will be able to gain an understanding of the difficulties which pupils encounter and will be able to devise more appropriate methods of teaching

INTELLIGENT TEACHING DEPENDENT ON AN ADEQUATE PSYCHOLOGY

Teaching is a complicated process because human nature is complicated It is a distinct disservice to education to try to describe mental life as made up of simple factors The psychology which pictures mental

processes as readily resolved into atoms of any kind is worse than useless, it is indefensibly misleading Mental processes are complex organized patterns of behavior Such is counting such are addition subtraction, and the other forms of thinking taught in arithmetic, such are the mental processes involved in the mastery of all lessons—in reading in looking at maps, in learning names and dates in history, and in acquiring the information taught in nature study It behooves all who try to guide mental life to gain as full knowledge as possible of the patterns involved in learning and in mature intellectual processes

CHAPTER III

COUNTING BY CHILDREN

TYPES OF CHILDREN INCLUDED IN EXPERIMENTS

This chapter will report the results of a series of measurements made with pupils in the University Elementary School of the University of Chicago by methods similar to those described in the foregoing chapter Twenty pupils were selected from each of the grades from Grade I to Grade VI inclusive and measurements were made of their abilities to count series of sounds and series of flashes of light In the case of each grade one half of the pupils were chosen from among those whom the teachers judged to be the most competent in number work and the other half from among those whom the teachers judged to be the least competent. In the aggregate therefore the tables in this chapter report on sixty highly competent pupils and sixty pupils who were less competent In no case was the pupil a complete failure in his grade The tests were given during the month of May the pupils were therefore near the period of promotion into the next higher grade

CONDITIONS OF EXPERIMENTATION

The experiment was conducted by testing each pupil individually A teacher accompanied each pupil and observed him while he counted the sounds or flashes of light It is possible therefore to report a number of observations made during the experiment. Furthermore the presence of the teacher tended to remove any fear or distraction The pupils talked freely with their teachers and made a number of comments which are illuminating

It was very difficult to secure during so long a series of measurements as was necessary for the 120 pupils absolute comparability of the series In the first place the number of sounds and flashes of light could not be perfectly controlled When the experimenter intended to give a series of seven sounds he sometimes missed by a single sound and gave a series of eight or six In the second place it was impossible to arrange in advance series of numbers that would be exactly comparable for all ages and for all stages of maturity It would have been possible to determine from the results secured from adults and reported in the last chapter the relative difficulty of counting various series as for example six sounds in one

series and fifteen sounds in another, but there is no guaranty that the weightings for mature persons would be the same as those for immature persons The effort to make absolutely exact comparisons was therefore abandoned The pupils were, however, given series which were sufficiently comparable to make possible a judgment as to the type and rate of progress from grade to grade

TABLE XI

TOTAL NUMBER OF ERRORS MADE BY TWENTY PUPILS IN THE FIFTH GRADE IN COUNTING SERIES OF SOUNDS

Pupil	Number of Sounds Given per Second						Total Number of Errors
	Two	Three	Four	Five	Six	Seven	
1	0	1	5	7	6	4	23
2	0	0	1	0	0	2	3
3	0	1	0	1	2	1	5
4	0	1	1	1	1	4	9
5	1	1	1	1	2	3	9
6	0	0	0	1	0	1	2
7	0	0	1	2	3	2	8
8	0	0	0	0	1	2	3
9	0	1	2	3	0	2	8
10	0	1	1	4	2	6	14
11	0	0	1	0	2	4	7
12	0	6	5	3	6	5	25
13	1	1	0	3	3	3	11
14	1	4	1	4	7	6	23
15	1	1	2	3	4	4	15
16	0	4	3	3	5	7	22
17	0	1	2	2	2	3	9
18	0	1	1	1	4	4	11
19	3	1	5	7	8	6	32
20	0	0	1	1	4	3	9
Total	7	25	32	48	62	74	

TYPICAL RESULTS SECURED WITH SERIES OF SOUNDS

In order to make clear the kind of data collected, Table XI is compiled from the results secured in the case of the pupils in the fifth grade when they were given series of sounds The first ten pupils, it should be remembered, were selected from the better half of the grade, Pupils 11 to 20, inclusive, were selected from the slow half of the grade It is evident from the number of errors recorded in the extreme right hand column that ability to count sounds is, in most cases, related to ability in number work as judged by the teacher

The totals in the line at the bottom of Table XI show clearly the effect of increasing the rate at which sounds are given The total possible

number of errors at each rate is 160. When the number of errors at any given rate is as low as seven, they may be regarded as purely accidental. Indeed, it may be said that, when the number of errors is less than one-fourth of the possible number, the ability of the pupils is fair. The point in this table at which the errors begin to be numerous enough to justify the statement that the pupils of this grade are deficient in counting is the point at which individual errors commonly equal or exceed three and the aggregate exceeds forty.

Marked individual differences are apparent in Table XI, as they are in Table II, which reports results for the adults It is quite possible to find pupils in the fifth grade who are equal in the ability measured to the more competent adults taking part in the experiment.

TABLE XII

TOTAL NUMBER OF ERRORS MADE BY TWENTY PUPILS IN EACH GRADE IN COUNTING SERIES OF SOUNDS

Grade	Number of Sounds Given per Second					
	Two	Three	Four	Five	Six	Seven
I	56	75	76	100	130	136
II	30	37	51	86	120	114
III	15	19	35	76	93	112
IV	10	17	36	54	75	91
V	7	25	32	48	62	74
VI	5	4	15	35	44	64

COMPARISON OF RESULTS FROM THE VARIOUS GRADES

Tables similar to Table XI were compiled for the different grades, an from these grade tables a general table was made up which compares th results in the various grades (Table XII)

The impressive fact exhibited in Table XII is the steady progress throughout the elementary grades in ability to count series of sounds No one who has examined this table can regard counting as a matter to be disposed of in any short series of drill exercises Throughout the period of elementary schooling, pupils improve in ability to count series of sounds First grade pupils know the number names and can use them with some degree of success when the number of sounds to be counted is small, but they can deal successfully only with sounds that are given at a very slow rate By the time pupils reach the third grade, they may be said to have become fully proficient in dealing with the slow rates By the time pupils reach the fifth or sixth grade, some of them exhibit a degree of proficiency comparable to that shown by adults

Table XIII exhibits the results secured when the pupils were shown flashes of light at various rates. Here, again, steady progress throughout the elementary school period is shown, but this progress is far behind that shown in Table XII for the counting of sounds. In this case, the fifth grade pupils were less proficient by far than were the adults. Apparently, development in this kind of counting is not only less complete for all individuals but much slower than is development in counting sounds.

TABLE XIII

TOTAL NUMBER OF ERRORS MADE BY TWENTY PUPILS IN EACH GRADE IN COUNTING SERIES OF FLASHES OF LIGHT

Grade	Number of Flashes of Light Given per Second			
	Two	*Three*	*Four*	*Five*
I	75	66	1[illegible]4	
II	51	9	121	145
III	33	65	96	134
IV	23	55	107	143
V	20	65	89	127
VI	9	47	91	122

OBSERVATIONS MADE DURING COUNTING

The teachers who accompanied the pupils were asked to observe especially whether the pupils made any movements of the lips or other parts of the body in counting. They were also asked to note any significant facts which would throw light on the methods of counting adopted by the pupils.

REPORT ON THE PUPILS IN THE FIRST AND SECOND GRADES

Nina Jacob, the teacher in charge of thirty two of the youngest pupils, twenty of whom were in the first grade and twelve in the second grade, reported as follows:

The accompanying observations were made while the children were doing the counting. On entering the room each child was shown the instruments on the table and told that one of them produced sounds and the other flashes of light. He was told that he was to count the sounds and the flashes of light. The machine in the adjoining room was then started so that the child might know what to expect. Next the purpose of the buzzer as a signal to get ready to count was explained; finally the counting was begun.

The majority of the children made several kinds of movements and used several methods of counting. The most common movements and the ones which appeared in almost every case throughout the test were lip movements.

The following are some of the observations made by the teacher and some of the significant remarks made by the pupils

Holding body tensely at attention
 Stopping to swallow in middle of count
 Holding lips in position to start counting
 Twisting fingers nervously in palm
Moving as in evident fatigue
 Restless twisting in chair
 Yawning
 "Are you through? I want to go " (Five year-old boy)
Uncertainty
 I think it was "
 "I don t know How many did you count?"
 "About three"
 "I couldn t count so fast
 "Eight! Seven I mean "
Confidence in accuracy
 I can count anything fast "
 "I know too much about counting to use my lips
 "I like to count to one hundred "
 I count to myself good and fast
 I think I can because I count a lot
Preference for flashes of light
 "Lights are easier They go faster
 I can count the lights better
 The lights are easier because they are slower
Preference for sounds
 The sounds are easier because they aren t so fast at the last
Additional individual reactions
 Noticing odd and even numbers— These are just the opposite of 'two' 'four' 'six' 'eight '
 Noticing colors in the lights— Oh look at the pretty color in the lights It s lavender"

Table XIV summarizes the observations made in the case of this group of pupils

REPORT ON THE PUPILS IN THE SECOND AND THIRD GRADES

Ada Polkinghorne, who was in charge of ten pupils in the second grade[1] and twenty pupils in the third grade, summarized her observations as follows

[1] The ten pupils in the second grade who were observed by Miss Polkinghorne raised the total number of second grade pupils to twenty two In order that the comparisons of the various grades might be based on like numbers of pupils the last two pupils in Miss Polkinghorne s second grade group were not included in making up the tables but observations are reported for the whole group

TABLE XIV

RESULTS OF OBSERVATIONS MADE WHILE THIRTY TWO FIRST- AND SECOND GRADE CHILDREN WERE COUNTING SERIES OF SOUNDS AND SERIES OF FLASHES OF LIGHT

	Age						Number of Boys	Number of Girls	Total Number of Children
	Five		Six		Seven				
	Boys	Girls	Boys	Girls	Boys	Girls			
Bodily movements accompanying counting									
Moved lips in silent counting	1	1	3	0	2	9	6	10	16
Counted aloud				1	1		1	1	3
Nodded head or tapped fingers in rhythm			1			1	1	1	3
Held body tensely at attention			1	2		1	1	3	4
Watched instruments intently				1	1	1	1	1	1
Moved as in evident fatigue	1			1	1	1	3	1	3
Methods of counting used									
No lip movement					1	3	1	3	4
Counted flashes of light much faster than they were given	1		1		1	1	4	2	6
Counted one beyond last sound or flash of light			1	2		1	1	4	5
Counted more than one beyond last sound or flash of light		1			3	1	3	1	4
Counted too slowly to keep up with flashes of light		1		9	1	1	1	4	5
Started to count on second sound or flash of light			1			3	1	1	3
Started to count on third sound or flash of light				1		1		1	3
Guessed				1	1		1	1	3
Started to count before sounds or flashes of light began				1				1	1
Held count in mind and finished the count after sounds ceased				1	1		1	1	4
Counted correctly but reported one less				1				1	2
Stopped too soon that is before flashes of light stopped			1					1	2
Rhythmic counting at rate of one count to two sounds						1		1	2
Added two or three groups in a series					1	1	1	1	1
Attitude toward results of counting									
Uncertainty	1		1	1		1	1	3	5
Confidence in accuracy			1		1	1	1	1	2
Knew answer given was wrong				1				1	1
Corrected answer				1				1	1
Preferences noted									
Preference for flashes of light		1		3				4	4
Preference for sounds			1				1		2
No preference stated	1		2	7	5	12	8	19	27
Additional individual reactions									
Noticed odd and even numbers of sounds in a series				1				1	1
Noticed colors in the flashes of light (lavender)				1				1	2
Number of pupils tested	1	1	3	10	5	12	9	23	32

Pupil 7, Grade V A, very good in arithmetic In the case of both the sounds and the lights, he whispered the numbers He thought the lights were harder to count

Pupil 8, Grade V A, very good in arithmetic She counted the lights without effort and thought that they were easier to count than the sounds In the case of the sounds, she moved her lips, or nodded her head, or did both In counting a few of the sounds, she tapped with her foot

Pupil 9, Grade V A, very good in arithmetic She moved her lips for all the sounds and lights She thought the sounds were easier to count

Pupil 10, Grade VA, very good in arithmetic. There were no outward signs of counting in the case of either the sounds or the lights

Pupil 11, Grade V B, weak in arithmetic He used his lips in counting all the sounds and lights He said, 'The lights were harder because they went faster, but I counted them in the same way"

Pupil 12, Grade V B, very good in arithmetic As he is quite deaf, he took a seat close to the apparatus In the case of the slowest sounds there was no definite lip movement, but there seemed to be a slight movement of the vocal organs After the first seven series of sounds he began to use his lips Sometimes he gave very indefinite answers, such as "It was either seven eight, or nine" He said, "I can always tell when the speed changes. We have had three speeds already' When he was counting the lights, he seemed very nervous He said, "The lights are harder to count than the sounds, they are in such a little place" He used his lips in counting all the lights

Pupil 13, Grade V B, an average pupil in arithmetic She used her lips in counting all the sounds and lights She said that the lights were harder to count than the sounds

Pupil 14, Grade V B, weak in arithmetic In the case of the first half of the sounds, he moved his lips He said that he moved his tongue to help him count, but there were no outward signs of such movement In counting the lights, he did not move his lips at all He thought the sounds were harder to count

Pupil 15, Grade V B, weak in arithmetic He moved his lips a little for all the sounds and lights. He made the following comments 'I counted with my tongue, I counted by five's, that is I grouped them, I liked the lights better because they were harder, I counted silently"

Pupil 16 Grade V B, very weak in arithmetic. She did not use her lips at all in counting When she was asked how she counted, she said "I just counted" "The lights were much harder, I couldn t see them as well"

Pupil 17, Grade V B, very weak in arithmetic She moved her lips in the case of both the sounds and the lights Before she became accustomed to counting the sounds, she also nodded her head The lights seemed harder for her to count than the sounds She said, 'All the sounds up to ten are in one group and the numbers above that in another group'

Pupil 18 Grade V B, very weak in arithmetic He counted in a very awkward way At first, he moved only his lips. Later he moved his lips head, and

hands Finally, his whole body swayed back and forth as the sounds were produced In counting the lights, he used only his lips, but his whispers were audible He said "I could count better when I used my lips and fingers" He seemed to think that the lights were easier to count

Pupil 19 Grade V B, very weak in arithmetic. At first, she nodded her head, tapped with her fingers and moved her lips when she was counting sounds As the sounds became more rapid, there were no outward signs of counting She said, 'I just looked at the lights I didn t have to use my lips" She thought the lights were harder to count She seemed very tense in her counting

Pupil 20 Grade V B, very weak in arithmetic She moved her lips slightly in the case of both the sounds and the lights She thought the lights were harder to count

REPORT ON THE PUPILS IN THE SIXTH GRADE

Adaline Sherman, who was in charge of the pupils in the sixth grade reported as follows

PUPIL 1

Good in number work Moved his lips in counting Said 'I m used to this I have a telegraph set at home ' It is harder to count the flashes My eyes get tired ' Before the last series of flashes of light he said It s easier to count now I m getting used to it"

PUPIL 2

Good in number work

Sounds —Her whole mouth moved as she counted Her lips showed that she caught the rhythm of the sounds and that she kept on counting one more than the number given but she always gave a number that was one less than the number she had counted to herself

Question 'How do you count these taps?

Answer 'I keep on counting

Question 'What do you do with thirteen fourteen fifteen etc?'

Answer 'It s hard to count thirteen fourteen fifteen but I do I could do the other way and get it easier Evidently she meant that she could group the sounds

Lights —In Series 1 with fourteen lights she said It s so funny I can hardly see it ' Her lip-movement became less noticeable as she watched the flashes of light more intently Later in the series she said I got through counting long after the lights stopped '

PUPIL 3

Good in number work Looked steadily at the table while counting the sounds During the first and second series there was a strong bodily motion forward This motion was less marked as the sounds and lights were speeded up In counting the sounds she said This is like counting telegraph poles going to Sawyer '

PUPIL 4

Good in number work He moved his heel as he counted the sounds and lights He was very self conscious and watched to see whether he was being observed On finding that he was he tried to stop the motion

Question How do you count these numbers?'

Answer "I just count one after the other"

PUPIL 5

Good in number work Lips moved

Question 'What do you do when you count these?'

Answer 'Just straight counting In getting into the teens I count 'one, two three, four,' again and in the twenties I say twenty-one, two, three, four, etc.''

PUPIL 6

Good in number work Very slight lip movement In counting the lights, he also nodded his head

Question "What do you do when you count these?'

Answer 'I just count them silently"

The lights hurt his eyes even in the first series There was more lip movement when he was counting the longer series of flashes of light In the last series of sounds and in Series 2, 3, and 4 of the lights he reported one number less each time

PUPIL 7

Good in number work *Observation in class work showed that he counted* when adding *Strong lip movement when counting the sounds Did not watch* the apparatus

Question 'What do you do when you count these?'

Answer I do it just the way he does If he grouped them I would'

Sounds Series 4—' Fifteen or sixteen I m not sure (The one before the last)

Sounds Series 5— Not sure'

Lights Series 1—He said *These are easier to count than the last series of sounds'*

Lights Series 4 and 5—He said, I m not sure each time nine lights were shown

PUPIL 8

Good in number work *Noticed the even progression of the numbers in* the first series of both the sounds and the lights

Question Have you any special way of counting the lights?

Answer Yes, I say one two *fourteen fifteen sixteen etc*

Lights Series 2 (eleven lights) — I got kind of mixed up on that There was a slight movement of the lips which was more noticeable as the numbers became larger and the flashes more rapid

Question (after Series 3 of the lights) "How are you counting?"

Answer Just counting in the regular way Sometimes I have to sort of guess. Some are even numbers and some are odd When they start with even numbers, they usually have an even one next

In Series 5 of the lights he became confused He said The lights are so bright and I m not used to counting lights.

PUPIL 9

Good in number work Showed marked concentration No perceptible movement of lips or body

Question How do you count these?

Answer I just count them I tap when I add It is easier to count than to add I'

PUPIL 10

Good in number work Lips moved He counted the sounds and lights just as they came

PUPIL 11

Only fair in number work A musical genius Practices a great deal with a metronome. Lip-movement

Question 'How do you count these?

Answer I take one two three four five one two three four five because it is easier that way

Question Do you have any trouble keeping track of the five s?

Answer Oh no I just feel it When I get about so many I just know it

He noticed the even progression in the first series

Lights —After the first number he said I find this harder After the second number he said This is harder to keep track of

Question Do you keep track of these in five s?

Answer No It s too hard

Used his fingers He was unusually poor in counting the lights

PUPIL 12

Poor in number work Slight occasional lip movement Said in answer to question I just say 'one two etc

Question Do you have any trouble saying the numbers to yourself?

Answer Yes sometimes I skip some and then I remember and count them in at the end If I skip three I count three more at the end

PUPIL 13

Fair in number work I just think the sounds I counted the first ones They kept going two more each time The first were even numbers then odd Then they were mixed up

Sounds, Series 5—Counted aloud at the same rate after the series of sounds was completed

Lights, Series 1—Said, 'These are even again It is hard to tell one light from another"

PUPIL 14

Fair in number work Pronounced lip movement

Question 'Have you any special way of counting these?'

Answer 'No just regular counting'

In the faster series he said 'Sometimes I make one more"

Lights—Much less lip-movement

PUPIL 15

Poor in number work

Question 'What do you do when you count these?'

Answer 'Just count them'

Question How do you count numbers larger than ten?'

Answer "I say, 'eleven twelve, thirteen etc.'"

Lights—Said, These are hard to count They go so fast '

PUPIL 16

Very poor in number work She sat very quietly and made little response to the questions She said that she 'just counted" Her lips moved when she counted the lights She said, 'These are harder than the sounds '

PUPIL 17

Very poor in number work Her lips moved while she was counting the sounds When she was counting the lights there was such slight lip-movement as to be almost imperceptible In answer to a question she said that she just counted Later she said I just think it When she was asked whether she grouped numbers in counting she said No it is easier to count one two three, four five etc.

PUPIL 18

Very poor in number work A very shy child She sat quietly and counted Her lip-movement showed that she counted in series.

PUPIL 19

Poor in number work Slight lip-movement He noticed the increase by two's in the first series

Question 'What do you do when you count the fast ones?"

Answer I just about always go one ahead so I take one off

Lights—Lip movement

Question (at the end of the first series) Do you find these harder?

Answer No easier

After the third series he was not certain that they were easier

PUPIL 20

Poor in number work No perceptible movement

Question "How do you count these signals?"

Answer "Just the way I count everything I think the numbers to myself"

Lights, Series 1 —"It's easier to count the lights"

Lights, Series 3 —Could not keep track of the numbers Said, "It's hard to get The lights go so fast I can hardly see them"

GRADUAL REDUCTION OF REACTIONS

The observations made by the teachers confirm the conclusion reached earlier that counting is an active process They also throw much light on the way in which individuals pass in the course of development from gross external reactions to much reduced internal reactions If the reports for the various grades are compared, it will be seen that lip movements and movements of the head or other parts of the body are gradually reduced as the pupils become more mature until these reactions become slight inner tendencies to articulate

The gradual reduction of counting to an inner mode of reaction is similar to the change which occurs during the mastery of all forms of language Like his counting, the child's reactions in naming objects and in ejaculating in the presence of stimulating situations are, at first, external and readily observed As the individual matures, he continues to use names in recognizing and classifying objects, and he continues to respond to stimulations, but his responses are more and more of an inner type

REACTION AN ESSENTIAL PHASE OF COUNTING

The conception of number ideas and of ideas in general which we are led to adopt on the basis of the foregoing discussion is of the highest importance for the student of education There is much educational writing which treats number ideas as though they were some kind of mental image As has been abundantly shown, number ideas are, in fact, more than images, they depend on the presence of reactions A child does not learn numbers by having them impressed on his organs of sense There is no such thing as a number sense Number is acquired only when there is a positive reaction One must respond in a definite way to each item of experience which is to be counted The definite positive response which one makes to each object counted is reduced to an inner reaction in the course of educational development but it continues to be a reaction

Such considerations as the foregoing help to explain why education takes time If number ideas were of such a nature that they could be

his use of the number names in the conventional order, he becomes aware of the point reached in a series of discriminating reactions Suppose, for example that the individual has reached the point where he reacts to the last object in the group by uttering the word 'eleven , he will in this case enjoy the double advantage of having an analytical knowledge of the group and of knowing that the number of objects in the group is more than ten or nine or eight and less than twelve or thirteen or fourteen

EVOLUTION OF EXACT QUANTITATIVE IDEAS

It takes a long time for children to learn the use of the valuable device for exact discrimination which is supplied in the series of number names that the race has evolved It took the race a long time to perfect the series of number names and to learn to use this series of words While evolving the number series the race was also developing a highly important general attitude namely the attitude of appreciation of the value of exact discrimination The savage does not understand how greatly his life would be enriched and its activities made more effective by exact discrimination and quantitative modes of thought Little by little the race has learned to count depending at first on the crudest and most external forms of reaction such as pointing successively with the various fingers or laying aside a pebble as an aid to each discrimination As the race has come to a recognition of the importance of a system of counting it has perfected its system It has also learned to impose on every individual the demand that descriptions be exact and the demand that all dealings be controlled by the general idea of exactness When a child comes into the modern civilized world he finds his elders equipped with an elaborate device for discriminating objects of all kinds and for recording their number The child like the race must gradually arrive at an appreciation of the value of exact discrimination No child sees at the beginning of life and for a long time thereafter the true value of a number series but society is so fully aware of the importance of counting that it does not leave the child to find out for himself why people count Society goes actively about inducing the child to use the number series Even with society stimulating the child by all the means at its command the acquisition of the number system is a slow process The acceptance and appreciation of the general idea of exactness by the child are even slower than the acquisition of the series of number names In fact in some cases pupils learn the number system in order to comply with the urgency of society and fail entirely to gain full personal appreciation of the real meaning of the series

CHAPTER IV

ANALYSIS OF INDIVIDUAL CASES

EXPERIMENTS WITH INTROSPECTIONS AND SPECIAL TESTS

Twenty graduate students of education acted as subjects in a series of counting experiments similar to those described in earlier chapters and kept introspective records throughout the experiments They also submitted to a number of supplementary tests The results are presented in this chapter in full detail for the purpose of bringing out the fact that there are important individual differences in the number ideas of mature persons

Table XV is similar to Table I in form and reports the full record of errors made by the twenty subjects in counting series of sounds Many of the facts shown in this table are quite similar to those discussed in earlier chapters Without commenting on the general findings for the whole group, we turn to a discussion of individual cases

VARIATIONS IN ACCURACY

Subject 42 showed complete accuracy through all the series of sounds up to and including that in which he was given six sounds per second He made some errors in the case of large numbers when the sounds were given at the rate of seven per second, and in the case of all series except the shortest he failed to count correctly when the rate was raised to eight sounds per second Subject 48 is evidently of a wholly different type A certain degree of inaccuracy was exhibited by this subject when the sounds were given at all the different rates except four per second Subject 42 reported in his introspections that he always counted by repeating the number names to himself and was clearly conscious of incipient movements of articulation Subject 48 was quite as explicit in stating that he was not aware of any counting movements He said that very often he fell into a rhythm of counting, the most common form of this rhythm being a grouping by three s Thus, he counted "one two, three, four, five, six, seven, eight nine," and so on

VARIATION WITH RESPECT TO INNER ARTICULATION

The other subjects who were quite certain that they said the number names to themselves are Subjects 43, 45, 47, 51, 55, and 58 The subjects

TABLE XV

ERRORS MADE BY TWENTY SUBJECTS IN COUNTING SERIES OF SOUNDS

Number of Sounds Given	Erroneous Numbers Reported by Twenty Subjects																				Number of Errors
	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	
Three sounds per second																					
8								9				7				6					3
14								15								10					2
22				23		21		23				21	23			17					6
5																4					1
10																8	11				2
Number of errors	0	0	0	1	0	1	0	3	0	0	0	2	1	0	0	5	1	0	0	0	
Four sounds per second																					
6												5				4	7				3
10																8	12				2
13	14			14										14		9	16				5
6																5	7		7		3
15	16				16	14						13		16		11	16				7
10	*											11				7	11				4
Number of errors	3	0	0	1	1	1	0	0	0	0	0	3	0	2	0	6	6	0	1	0	
Five sounds per second																					
10																8	12				2
12													11			11	13				3
6												5				5	7		7		4
11				10				12					12			9	12				5
16	17					14			15		15	15	14	17		14				15	9
19	20			20		15		18	17		17	16	18			16	18	18	18	17	13
7						6			8			6				6	8		8		6
Number of errors	2	0	0	2	0	3	0	2	3	0	2	4	4	1	0	7	6	1	3	2	

* No number reported

who were explicit in stating that they did not say the names to themselves are Subjects 50 53, and 60 The other subjects seemed to have a mixed practice

There is a certain generic resemblance among the records of the subjects who said the number names to themselves In general they were accurate up to a fairly well defined point and then rapidly fell behind the sounds and were unable to count accurately The subjects who did not say the number names to themselves were less able to give clear statements of what they did They frequently spoke somewhat vaguely about grouping the sounds Sometimes they spoke of visual methods of counting

VISUAL METHODS OF COUNTING

In this latter connection it is interesting to note that Subjects 50 and 60 showed in the records made with flashes of light as reported in Table XVI complete accuracy at first and an abrupt falling off at a certain point In other words there is some evidence in the later table that they reacted to visual experiences in a way that differed from their method of reaction to sounds

UNDERCOUNTING AND OVERCOUNTING

Another very striking type of individual difference is exhibited in Table XVII by Subjects 56 and 57 Both were poor in counting Subject 56 invariably reported a number which was too small while Subject 57 in somewhat more than half the cases reported a number which was too large The introspections of both subjects showed that they were greatly confused Subject 57 noted explicitly that he did not always know whether he stopped counting at the right time

The difference between the two subjects seems to arise from the differences in procedure It seems that Subject 56 was unable to keep up with the objective series His subjective series of number names was not fluent enough he therefore frequently dropped behind the objective series When the numbers were large he sometimes dropped behind more than one The errors of Subject 57 were of a wholly different type In his case the subjective series often absorbed so much of his attention that he did not recognize immediately the arrival of the end of the objective series and continued counting after the objective series stopped He sometimes undercounted especially in the case of large numbers but in the main his errors were of the other type The subjects who predominantly undercounted are Subjects 42 43 46 49 50 51 52 53 55 56 and 58 Those who predominantly overcounted are Subjects 41 45 57 and 59

TABLE XVI—*Continued*

Number of Flashes of Light Given	Erroneous Numbers Reported by Twenty Subjects																				Number of Errors
	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	
Four flashes per second																					
5	4			4		4	4				4	4		4		4				4	9
4						5				7	3			3			5			3	6
8	6			6	7	6	7			7	6	6	7	5		7		7	7	6	14
5	4		4	4	4	4					4			4		4				4	9
18	13		12	16	15	14	16		17	16	13	13	16	10		13	14	16	15	14	17
10	8		8	7	8	7	8		8	9	8	6	8	5	9	8	8	9	9	7	18
12	11		10	8	10	9	10		11	11	9	9	10	8		*	10	10	11	9	17
10	8		9	7	8	9	9			9	8	8	9	6			9	9	9	8	15
21	19		18	17	18	16	18	*	17	18	14	16	19	10	18	16	18	16	18	15	19
Number of errors	8	0	6	8	7	9	7	1	4	7	9	7	6	9	2	7	6	6	6	9	
Five flashes per second																					
5	4			4	4	4		4			4		4	4			6	4	7	4	12
4			3	3		5				3	3			3		6	5	3	6	3	11
10	9		8	6	9	9	8		9	8	7	8		6	9			8	13	7	15
10			7	7	8		8		9	7	7	8	9	5	9			8	13	7	14
5	4		4	3	4	6	4			4	4		4	3	4			4	6	4	14
Number of errors	3	0	4	5	4	4	3	1	2	4	5	2	3	5	3	1	2	5	5	5	
Total number of errors	12	3	11	16	13	13	12	4	6	11	16	16	9	19	5	19	11	14	11	17	

* No number reported.

A few undercounted in the rapid series and overcounted in the slow; such are Subjects 44, 48, 54, and 57.

Evidently the lapse of attention which resulted in the overlooking of one of the sounds is the more common source of error. The existence of the other type of case, however, is highly significant The fact that there are two wholly different methods of controlling attention shows that, as individuals develop in childhood and acquire the ability to count, their histories are very different. The type of attention which leads to overcounting must result from overemphasis on the subjective series. The overemphasis does not seem in some of the cases at least to have been due to lack of fluency.

VARIATIONS IN COUNTING FLASHES OF LIGHT

Table XVI supplies evidence in partial confirmation of the hypothesis that overcounting is not due to lack of fluency This table shows the records of the twenty subjects in counting flashes of light. Of the four subjects who overcounted sounds, two, namely, Subjects 41 and 45, changed their methods of counting and undercounted, Subject 42 changed to overcounting

The changes in type of error when the impressions to be counted vary in character suggest that the modes of development in the two sensory spheres are, in some cases, entirely different Subjects 41 and 45 apparently found counting flashes of light especially difficult, and Subject 42 apparently found counting flashes of light especially easy The ease or difficulty of counting may be due to the amount of experience in this sphere, or it may be due to general interest in visual objects or lack of such interest

Whatever the explanation, it is quite evident that the processes of counting in those cases where there is overcounting are different from the processes of counting where there is undercounting There is danger that anyone who gives the results of the experiments a mere statistical treatment will be misled by the fact that the same series of number names is used by individuals who overcount and those who undercount The assumption that the psychological processes are identical in these two cases fails to recognize the full importance of the psychological contrast which is indicated by the results of the experiments reported

COUNTING WITH SIMPLE NUMBER NAMES

The introspective records explain some of the particular points shown in Table XV where the subjects seemed to depart radically from their own

to put him at the bottom in both cases, Subject 47 stood very high in both cases

In the presence of such facts one grows skeptical of the value of arithmetic tests as the sole means of guiding teachers in dealing with pupils Equal numbers of errors do not always indicate the presence of like difficulties Aggregate scores frequently leave the teacher without knowledge of the individual difficulties which must be overcome if pupils are to be helped to correct their deficiencies Tests locate difficulties and for that reason are valuable instruments of diagnosis When an error has been located its type must be discovered by some kind of detailed analysis

Nor is it merely in cases where errors appear that analysis is necessary if one is to evaluate properly scores secured in tests Ample evidence has been supplied that the positive methods by which pupils succeed in dealing with numbers are not alike even when the degrees of success achieved are entirely comparable One pupil may be rapid in counting because he is rapid in articulation another may be rapid because he reduces his articulations to a minimum The ability to transfer attention rapidly from the objects which are being counted to the subjective series may be a third explanation of rapid counting If such a variety of explanations is found to be justified in dealing with the simple process of counting how much more must it be true that the complex processes of arithmetical combination involve procedures of very different types even when they yield results which are alike?

SCORES ON THE CLEVELAND SURVEY ARITHMETIC TEST

In order to study the relation of the results thus far presented in this chapter and the ability of the twenty subjects as measured by a standard arithmetic test the first seven sections of the Cleveland Survey Arithmetic Test were given to the subjects who participated in the experiment The results are presented in Table XVII The standard eighth grade scores for these sections of the test as reported by Counts [1] are included in the last column of this table

Approximately two-thirds of all the scores shown in Table XVII are above the eighth grade standard The subjects who were consistently low are Subjects 43 46 (except in Section D) 52 53 and 59 (except in Section C) Neither exception noted is large If we turn to Table XV we find

[1] George S Counts *Arithmetic Tests and Studies in the Psychology of Arithmetic* p 22 Supplementary Educational Monographs No 4 Chicago University of Chicago Press 1917

TABLE XVII

SCORES MADE BY TWENTY SUBJECTS ON THE FIRST SEVEN SECTIONS OF THE CLEVELAND SURVEY ARITHMETIC TEST

Section of Test	Subject 41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	Standard Eighth Grade Score
A	34	40	22	30	37	28	46	43	29	55	36	25	23	32	38	36	35	42	22	36	28 9
B	30	30	22	25	36	17	34	37	33	50	37	18	25	25	37	35	31	31	16	32	25 8
C	16	13	18	24	24	17	33	24	21	42	25	19	17	22	30	27	28	25	24	14	19 9
D	19	13	21	22	25	25	27	28	23	49	29	21	21	17	24	30	31	19	22	25	22 8
Total	99	96	83	101	122	87	140	132	106	196	127	83	86	96	129	128	125	117	84	107	
E	6	10	7	8	8	6	10	11	6	16	7	6	6	5	8	9	8	9	5	10	8 0
F	10	17	8	16	13	9	14	16	12	19	18	10	9	12	15	14	16	12	10	14	10 6
G	5	11	6	8	7	5	10	9	4	16	8	5	6	9	8	8	7	6	6	8	6 7
Total	21	38	21	32	28	20	34	36	22	51	33	21	21	26	31	31	31	27	21	32	

radically changing the character of his ability to count. Since the ability to count flashes of light was, in general, less fully developed than the ability to count sounds, a combination of sounds and flashes of light was given, thus providing the subjects with a means of improving their discrimination of flashes of light The combined series of impressions was first given to the group of twenty subjects The rate at which the combined sounds and flashes of light were given was a rate at which all the subjects showed a high degree of deficiency in counting flashes of light and a fair degree of proficiency in counting sounds, namely, five per second Table XVIII presents the results. It should be noted that a relatively large number of stimulations was used in all but one case

The number of errors reported in Table XVIII, while somewhat greater than the number of errors reported in the third section of Table XV, is distinctly less than the number of errors reported in the last section of Table XVI

TABLE XVIII

ERRORS MADE BY TWENTY SUBJECTS IN COUNTING SERIES OF SIMULTANEOUS SOUNDS AND FLASHES OF LIGHT GIVEN AT THE RATE OF FIVE PER SECOND

Number of Sounds and Flashes of Light Given	Erroneous Numbers Reported by Twenty Subjects																				Number of Errors
	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	
5	6				4	4						4				*			6		6
12						11				11	10	10			11	10					6
18	17		15	17		15			17	17	15	14			17	15	16	16		17	13
20				19		18			19	19	16	16	17	18	19	16	17	19	19	18	14
12			11	13	13				13			10			13	11				11	8
Number of errors	2	0	2	3	2	4	0	0	3	3	3	5	1	1	4	5	2	2	2	3	

* No number reported

were given at each sitting series of combined sounds and flashes of light In general, nothing but combined series were given Various rates were

TABLE XIX

ERRORS MADE BY FOUR SUBJECTS IN COUNTING SERIES OF FLASHES OF LIGHT BEFORE AND AFTER TRAINING

NUMBER OF FLASHES OF LIGHT GIVEN	ERRONEOUS NUMBERS REPORTED BY FOUR SUBJECTS				NUMBER OF ERRORS
	Subject 44	Subject 51	Subject 58	Subject 60	
Before training					
Five flashes per second					
5	4	4	4	4	4
4	3	3	3	3	4
10	6	7	8	7	4
10	7	7	8	7	4
5	3	4	4	4	4
Number of errors	5	5	5	5	
After five half hours of training					
Five flashes per second					
5		4			1
5					0
8	7	7			2
8		6	7		2
10	9	8			2
12	11	11	11		3
14	10	12	13	13	4
Number of errors	4	6	3	1	
After ten half hours of training					
Five flashes per second					
5					0
8	7	7			2
10		9			1
14	12	10			2
Number of errors	2	3	0	0	
Six flashes per second					
5					0
9	8	7			2
10		9			1
12	11	10	10		3
Number of errors	2	3	1	0	

used so as to avoid fixed habits of attention on a single rate At the end of the fifth practice period and at the end of the tenth practice period tests were given with flashes of light only The results are presented in Table

XIX In order that comparison may be facilitated this table includes the section of Table XVI which shows the number of errors made by the four subjects in counting flashes of light given at the rate of five per second before they had any training

In interpreting Table XIX, one should keep in mind the fact that the final accuracy of the various subjects in counting flashes of light is limited to their ability to count sounds One should consult Table XV therefore for what may be called the maximum possibility of improvement The only expectation that can be entertained above that established by Table XV arises from the possibility that the subjects might have improved during the practice series in their discrimination of sounds as well as in their discrimination of flashes of light.

Table XIX gives convincing proof that the training given in counting combined series of sounds and flashes of light produced very marked changes in three of the subjects and some change in the fourth subject

DEGREE OF PERMANENCE OF TRAINING

Unfortunately the permanence of these changes could not be ascertained after any protracted period of time Tests were given to Subjects 44 and 60 four weeks after the training was completed with the results shown in Table XX Evidently the changes shown in Table XIX were not permanently effective in full degree However marked effects of training still appeared as will be seen by comparing Table XX with the first section of Table XIX

TRAINING IN DISCRIMINATION

The changes reported in Tables XIX and XX cannot be attributed to any change in the mastery of the series of number names Ten half hours of practice in the case of an adult will not change his ability to use the number names with which he has long been familiar What must have taken place is an improvement in discriminating flashes of light The subjects learned to recognize the individual flashes of light and for this reason were better able to apply the series of number names—what we have called the subjective series —to the objective series of flashes of light

ANALYSIS OF THE COUNTING PROCESS

The counting process is thus analyzed into three distinct phases (1) possession of a subjective series of number names or number reactions (2) discrimination of the individual items of an objective series (3) application of a number name to each of the items in the objective series so that there is a one-to-one correspondence between the two series

The various phases of the counting process are capable of detachment from one another. Ability to repeat the series of number names may be improved through practice even when an individual is not attempting to count anything. The conventional series of number names which has been developed in the course of history is so elaborate that a child has to

TABLE XX

ERRORS MADE BY TWO SUBJECTS IN COUNTING SERIES OF FLASHES OF LIGHT FOUR WEEKS AFTER THE TRAINING WAS COMPLETED

Number of Flashes of Light Given	Erroneous Numbers Reported by Two Subjects		Number of Errors
	Subject 44	Subject 60	
Five flashes per second			
5			0
4			0
8			0
12	10	11	2
10	9		1
5			0
Number of errors	2	1	
Six flashes per second			
5			0
4			0
7			0
9	8	8	2
12	10	11	2
12	11	11	2
8	7		1
Number of errors	4	3	

devote much mental energy to acquiring the system itself. In like manner it is possible to discriminate a number of flashes of light or a series of auditory or tactual impressions as a succession of experiences without counting them. Here again the experience is sufficiently complex to require training. We may go outside the field of this experimental inquiry and take an example from ordinary life. If one stands before a garden of flowers one may discriminate the various blossoms, paying attention to their forms and colors but making no effort to count them.

Even after a person fully learns the series of number names and after he has reached a stage of mature discrimination of a certain group of

experiences counting the objects discriminated does not necessarily follow The two phases of experience may exist as possibilities in the mind but one may not call up the other For example one looks at a row of books on the library shelf and distinguishes them One could count them if one were so disposed but the occasion may call for an attitude wholly different from the counting attitude discrimination is then of a non arithmetical type

TEACHING AS A COMPLEX PROCESS

The fact that there are various stages or aspects of counting is the source of much of the difficulty which confronts the teacher in organizing the educational process Usually teachers teach children the series of number names in connection with a series of objects which are to be enumerated For example, a pupil is asked to count the members of the class When the pupil stumbles in his counting the teacher has to decide what is the difficulty Sometimes the number name is not recalled This is commonly the case in the lowest grades Sometimes the pupil is lost because he becomes confused in the effort to recognize the point at which he last looked This means that for the moment he has become confused in his discriminations An adult will cultivate sympathy with a child who gets lost in a series of impressions which are difficult to discriminate if he will try to count the letters on a printed page without pointing to the individual letters At times it will be noted that the letters are exceedingly difficult to distinguish from one another They seem to run together The difficulty in this case is not in the subjective series of number names it is in the process of discrimination

The child who is at one and the same time acquiring the number names and learning to distinguish objects has a third problem namely that of matching the two sets of experiences to each other In order to be able to count correctly he must make the transition from a point in the number name series to a point in the discriminated external series and must then come back to the number name series and so on In other words he must perform the mental feat of moving back and forth between two series of experiences which are different in character thus establishing a relation in his mind between the two

TRUE COUNTING A RELATIONAL PROCESS

Recognizing the relation between the number name series and the series of discriminated objective items is different from repeating the number names and also different from discriminating objects it is in an important sense a new fact of mental life It is not however a factor

capable of independent consideration, as are the series of number names and the series of discriminated objects It is a relational form of consciousness and, as such, can be thought of as a mental pattern which unites the two factors, number names and discriminated objects, holding these two factors together in a certain form of organized experience

TRUE COUNTING A PATTERN OF CONSCIOUSNESS

The conception that certain school processes develop patterns of organized experience rather than new factors or items of consciousness is a very fruitful conception for educators to understand When this type of psychological explanation of mental life is adopted, one begins to think of ways in which experience is organized rather than merely of factors or items Mental life is recognized as a growing organized whole

We shall have occasion in a later chapter to come back to a further elaboration of this idea It clears up some of the difficulties which we encountered when we found that some people are very competent in counting but rank low in tests in addition, subtraction, multiplication, and division

TRAINING AS AN ELEMENT IN EXPLAINING DIFFERENCES IN ARITHMETICAL ABILITIES

Our experiments have made it clear that training is a matter of vital importance in determining an individual s proficiency in counting They have shown that even short time training radically affects so fundamental an ability as that involved in counting flashes of light The hypothesis that all arithmetic scores are determined in character by native ability is thus shown to be invalid The evidence shows that organization of experience is a matter of prime importance in explaining an individual s number experiences and we are stimulated to undertake a further analysis of the mental processes involved in addition, subtraction, multiplication, and division

CHAPTER V

LARGE NUMBERS AND NUMBER COMBINATIONS

SPECIAL CHARACTER OF THE NUMBER NAMES BEYOND "TEN"

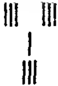
It was shown in preceding chapters that long series of impressions are more difficult to count than are short series Through experimental tests, this fact was also connected with the complex form of the number names beyond "ten"

The character of the number names cannot be accepted as a final explanation of the difficulty involved in counting long series The relatively cumbersome character of the number names beyond "ten" must itself be explained These names are due to the methods of operation of the minds which devised them Our inquiry must, accordingly, be extended until *it arrives at a fundamental explanation of the reason why the* number names beyond "ten" are complex

ATTENTION HAS A LIMITED SPAN

It is a fact well known in ordinary experience that the mind is not able to deal directly with a large number of objects Whenever a large number of objects is presented, the observer is forced to adopt indirect methods of dealing with the group What is meant by this statement can be readily demonstrated by asking the reader to inspect the following group of lines.

The lines are too numerous to be recognized fully and clearly If we contrast this first group of lines with a second group of lines equal in number but somewhat differently arranged, we note at once a marked contrast in ease of apprehension The following group of lines can be recognized much more readily than the first group presented



It is the arrangement in smaller groups and the separation of the smaller groups from one another which make direct recognition relatively easy We may conclude from this demonstration that there is a natural tendency *to break up large groups of objects into small groups* The mind resorts to analysis because it is incapable of apprehending a large group

SIMPLE TALLIES AS AIDS IN COUNTING

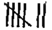
There is a second method of dealing with large groups of objects, that is *the method of substituting simple tallies for the objects* Let the reader inspect the following group of geometrical forms

It will be seen at once that the most obvious result of the inspection is a recognition of the contrasts in form One does not at first think of counting the figures If however a line is substituted for each figure, counting is both suggested and made easy Tallies or lines substituted for the figures are as follows

|||||||

GROUPING TALLIES IN UNITS OF HIGHER ORDER

In this case as in the case discussed in the earlier paragraphs the tallies are somewhat too numerous for immediate apprehension It is customary therefore to group the tallies as follows



With such a grouping the results of counting the geometrical figures are reduced to their simplest terms

Counting of every type is in a sense a method of dealing with objects which have become so numerous and so complicated in themselves as to pass beyond the possibility of complete and ready recognition Our analysis shows that *the way in which counting helps the mind to compass a large number of objects is by resolving the large group either directly or indirectly into smaller groups Indirect grouping is effected by substituting tallies for the full impressions of the objects*

of recognition which is possible in dealing with small groups has entirely disappeared

INDIRECT AND DIRECT NUMBER EXPERIENCES

We are thus led to a recognition of a very fundamental fact. Arithmetic includes certain direct forms of experience, such as observing four objects and clearly distinguishing them from five objects Arithmetic includes, also, a vast array of number facts which are incapable of direct apprehension The only way in which the mind can deal with the indirect facts is through the use of tallies or through the analysis of large numbers into small groups

METHODS OF THINKING BY MEANS OF TALLIES

The system of number which we conventionally employ has resolved all large numbers into manageable groups of ten Beyond ten, numbers are apprehended only indirectly For example, when one thinks of the difference between eighty six and eighty seven, one uses as the point of departure for one's understanding of the contrast not the beginning of the number series but eighty Eighty six is thought of as smaller than eighty seven because it is fairly easy to think of six as smaller than seven

SYNTHETIC ARITHMETICAL PROCESSES

The counterpart of the analytical process which resolves large numbers into groups of ten is the synthetic process by which the mind builds up large numbers The simplest exhibition of synthesis appears in counting when 1 is added to any given number Thus, when 1 is added to 4 the result is the larger group of 5 when 1 is added to 95 the result is 96

Increasing a group by one is, however, a slow process and there immediately arises a demand for some method of combining groups each one of which is larger than unity The demand for a method of combination which shall be adequate when the numbers to be synthesized are large leads away from simple counting and requires the invention of a method of intellectual procedure which is doubly indirect Suppose, for example, that one has twenty five objects in one hand and nine in the other How can one combine the two groups and know with precision the magnitude of the result? A primitive and simple method of procedure is to resolve the group of nine into nine single objects and add unit after unit to the twenty five until the total is reached

Combination by counting is undoubtedly the first and most natural procedure in uniting large numbers There is much evidence in such a number system as the Roman numerals that this procedure of unit addi

tions was one of the chief methods of combination The forms VII, VIII, XII, and XIII supply examples of such a process On the other hand, the Roman system, with its V, X, L, C, D, and M, evidently was not limited to mere increases by units There came a time when number combinations like XV and LX could be made freely without going back to unity to build up the larger number

COMBINATION THROUGH THE USE OF REMEMBERED RULES

The Roman examples of addition by the use of larger groups bring us to the point where we can understand the statement that the usual method of combining numbers depends not on direct procedures but on the use of certain formulas of combination which are even more indirect than is the knowledge of large numbers Let us illustrate by such an example as 7 combined with 6 The trained individual does not count He does not break up either number into smaller groups He is trained in the use of a rule of combination, and he arrives at the result 13 without detailed analysis of the numbers So completely is this process dependent on the possession of an indirect formula that it is usually impossible for the trained adult to tell how he made the combination In fact, he does not make the combination each time it recurs he escapes the necessity of performing the addition in detail by following a formula

Arithmetic is made up of a great number of rules of procedure which are of the type illustrated in the example of 7 combined with 6 Indeed if one examines a textbook in arithmetic one finds that very slight attention is paid to counting If one goes to a classroom where instruction is being given in arithmetic, one frequently hears the pupils warned against counting These facts supply clear evidence that the aim of instruction in arithmetic is to raise the level of thinking to the point where indirect processes will preponderate and as little attention as possible will be given to the simplest method of combining numbers that is counting

IMPROVEMENT IN COUNTING WITHOUT INSTRUCTION

The absence of instruction in counting in the elementary school is not paralleled by a lack of development on the part of pupils in ability to count It was shown in chapter iii that throughout the period of elementary education pupils improve steadily in ability to count One is tempted to attribute the low degree of ability exhibited in counting certain unfamiliar experiences such as flashes of light to the failure of the school to give instruction in the use of the number series in various spheres of experience Be this as it may, the experiments with children show that counting or the direct use of number continues and steadily improves in

spite of the fact that the school devotes its attention to indirect processes of combination.

EARLIER METHOD OF TEACHING FORMULAS OF COMBINATION

The method of teaching the rules of number combinations which was common two generations ago was to require the pupils to commit to memory addition and multiplication tables and from these to derive subtraction and division tables. The tables thus memorized were arranged in such a way as to approximate counting. For example, the addition table was made up of such items as 2+2, 2+3, and 2+4, each item differing from its predecessor by the addition of 1. The multiplication table dealt with more complex combinations but progressed in regular steps. These tables were highly systematic and were intended to give the pupil an idea of the regularity of number relations. The tables were learned before any applications to concrete situations were attempted. The tables may be described as tables of pure number relations.

PRESENT DAY METHOD LESS SYSTEMATIC AND MORE CONCRETE

The method of teaching the formulas of number combinations which is in common use today is much less systematic than was the earlier method of teaching by tables. At the present time all pure number relations are taught after a discussion of concrete situations.

If we take as an illustration one of the widely used textbooks in arithmetic[1] we find that this book begins its first lesson by treating certain units of space. The pupils are asked to make rules two inches long, four inches long, and so on. They are then asked to add together two inches and two inches, four inches and two inches, and so forth.

It is not until page 5 is reached that examples in pure number relations are presented, and here there is no hint in the arrangement of the examples that the number series and its combinations are systematic and related to counting. Some of the examples in addition found on page 5 are as follows:

2	3	4	2	1	4	1	2	4	4
3	4	2	1	6	3	5	7	4	1

After requiring a few exercises in counting, a second commonly used text[2] introduces an easy way of choosing sides in playing games, the pupils

[1] Edward Lee Thorndike, *The Thorndike Arithmetics*, Book One. Chicago: Rand McNally & Co., 1924.

[2] Thomas Alexander and Charles Madison Sarratt, *The Alexander-Sarratt Arithmetics*, Primary Book. Richmond, Virginia: Johnson Publishing Co., 1924.

are numbered and those with even numbers are on one side while those with odd numbers are on the other side The book passes from this to the use of cards marked as dominoes Referring to the domino cards the book presents such statements as 2 and 1 are 3 and '3 and 1 are 4 It then introduces an explicit statement about the plus sign and its verbal equivalents and on page 6 presents such examples as the following ' How many are 6 and 1? $6+1=?$ On page 7 is a series of pure number problems like those quoted from the first book

The concrete situations which are used in these textbooks to introduce the notion of combination cannot be said to furnish opportunities for the application of arithmetical formulas because the description of these situations precedes the statement of the formulas The situations are introduced rather for the purpose of supplying material which will tend to justify in the minds of pupils the learning of number combinations By thinking of these situations and of the number combinations which they demand the pupils are supposed to become interested in the rules which hold for the combination of numbers when applied to any objects

Even if one accepts without question the method of introducing the pure number combinations by a discussion of concrete situations it is not evident that there is any gain in mixing the combinations in such irregular order as to obscure the systematic character of number relations It seems to be the effort of modern textbook makers to get as far away as possible from the older systematic drill In order to discard the older methods entirely and finally they have gone to the extreme of abandoning all reference to the systematic regularity of the combination tables

A curious result has followed this abandonment of systematic drill Modern textbook makers and proponents of the present day unsystematic drill have found it necessary to secure completeness by some method consequently they have devoted a great deal of energy to the discovery and enumeration of all the possible combinations that can arise in addition and in the other fundamental operations They have also vigorously insisted that each and every combination receive its share of attention in the course in arithmetic In other words it is recognized that there are certain rules of combination in addition and in the other fundamental operations and that all these must be mastered even if they are not learned systematically The older textbooks pushed the number system into the foreground the modern textbooks push the number system as a system into the background but attempt to cover all that was taught in the earlier systematic tables

THEORY OF SPECIFIC TRAINING AS THE SOLE FORM OF EDUCATION

The method of organizing arithmetic which is shown by these examples to be the method in vogue at the present time is defended by its advocates on the psychological theory that all learning is specific and that the mind is made up of a number of detached special abilities Each combination is regarded by such theorists as a separate factor or bond

THEORY OF GENERALIZED SYSTEMATIC TRAINING

There is another view of the nature of consciousness namely, that the mind generalizes experiences and organizes its operations into certain *systematic schemes The mind is thought of as capable of formulating* a general idea of addition and a general idea of subtraction and general *ideas of the other arithmetical operations* The trained individual is thought of as being able to work out for himself, according to a systematic general plan, any scheme of number combination of which he understands the general principle It is this latter view which is clearly supported by all the *evidence presented in this monograph with regard to patterns of* mental organization

If pupils do not have presented to them the fact that the number system is a coherent, orderly scheme of thinking they are certainly deprived of one of the *most important lessons* in *scientific generalization* which the mathematical sciences can teach If arithmetic is taught as a series of wholly detached special rules of combination, the pupil is deprived of the most valuable aid to memory and scientific thinking which the mind possesses, namely, the idea of a general system in number

CONFUSION FROM VARIATIONS IN VERBAL FORMS

As the matter stands at the present time many of the textbooks not only omit systematic number tables but confuse the situation further by giving the pupil an overwhelming variety of concrete situations in which the difficulties are not chiefly arithmetical they are chiefly difficulties which arise in the course of the pupil s effort to understand the descriptive accounts of the situations

The way in which the descriptions of concrete situations complicate matters can be illustrated as follows A textbook designed for use in the second and third grades presents on page 6 a series of problems which are in reality problems in addition The questions in which the problems end are variously worded ' How much did he spend for both? ' ' How much do both together cost? ' ' How long are [they] together? ' ' How many cents were in the bank then? ' The words in these four questions which im

ply addition are "both," "both together," "together," and "were in the bank." In order to recognize that these phrases have a common element, the pupil must be able to look through their differences in form and to recognize the fact that all of them mean "combine by addition" The word "addition" is, however, avoided in each of the problems in favor of what may be called a "popular synonym" The ability to recognize all these phrases as having a common import is what is known as the power of abstraction Abstraction is one of the difficult steps in reasoning When it is remembered that number is itself abstract and that the process of combination is an indirect process, it will be seen how confusing is the further demand for abstraction which is imposed on the pupil by the requirement that he recognize a particular kind of combination even when it is described in various ways

DIFFICULTIES IN ARITHMETIC NOT ADEQUATELY UNDERSTOOD BY TEACHERS

It cannot be too often reiterated that arithmetic has proved to be the most difficult of the school subjects While the methods of teaching reading have so far improved that the great majority of pupils are successfully taught, the methods of teaching arithmetic are for some reason so ineffective that a large percentage of pupils fail The explanation of this fact is undoubtedly to be found in the inadequate analysis of the mental processes involved in learning numbers and number combinations The earlier chapters of this monograph have presented facts which prove that counting is a complex process which matures very slowly We turn now to a detailed demonstration of the truth which is stated in an introductory way in the foregoing paragraphs, namely, that the number combinations to which pupils are introduced in arithmetic courses are difficult because of the many complexities which result from the modes of describing the situations in which numbers are to be used

ANALYSIS OF TEXTBOOKS

For the purposes of this study, an analysis was made of four series of textbooks which are extensively used in elementary schools[1] The ques

[1] *a*) Thomas Alexander and Charles Madison Sarratt *The Alexander-Sarratt Arithmetics* Primary Book pp x+318 Intermediate Book, pp x+412 Richmond, Virginia Johnson Publishing Co 1924

b) Charles F Chadsey and James H Smith *Efficiency Arithmetic* Primary pp vi+282 Intermediate pp vi+314 New York Mentzer Bush & Co 1920

c) John C Stone *The Stone Arithmetic* Primary pp xiv+306 Intermediate pp xiv+322 Chicago Benj H Sanborn & Co 1925

d) Edward Lee Thorndike *The Thorndike Arithmetics* Book One pp xvi+268, Book Two pp xvi+286 Chicago Rand McNally & Co 1924

CONFUSION FROM GENERAL ARITHMETICAL PHASES OF PROBLEMS

Pupils not infrequently meet one of the first of their difficulties in arithmetic because of the general character of the words ' how many These words are usually used first in connection with problems in addition It often happens thereafter that the pupil tries to answer all questions which include the words how many by adding He overlooks the qualifying words which sometimes turn the question into one requiring subtraction or multiplication

CONFUSION FROM UNFAMILIAR SITUATIONS

Leaving the arithmetical side of problems to be taken up later we may well comment somewhat further on what has been called the situation phase of arithmetic problems A striking illustration of the difficulty which pupils experience in understanding the situations involved in arithmetic problems was supplied by a fourth grade teacher The problem given in the particular instance reported was one which was intended to correlate arithmetic with civics The problem stated that in a certain precinct a certain number of ballots had been cast and that in another precinct another number of ballots had been cast The question was in the familiar form How many were cast? One of the pupils was unfortunately ignorant in matters civic and did not know how ballots are cast Possibly there was a vague notion about the word cast derived from an early encounter with such a phrase as cast out At all events the pupil subtracted rather than added The mistake in this case was due to failure to understand the material or situation side of the problem

VARIETY IN PROBLEMS IN ADDITION

When both the arithmetical side and the situation side of problems in addition are considered it is possible to find a large array of problems which can legitimately be described as different The exact number of types distinguished will depend somewhat on the fineness with which one discriminates between situations which resemble one another but are not exactly alike For example the problems may properly be thought of as different on the situation side when the pupil is asked to purchase several books and when he is asked to purchase food for luncheon There can be little doubt that the pupil s familiarity with the two situations may be very different and that the mental imagery and the personal interest involved are certainly different On the arithmetical side a distinction can properly be drawn between the imperative form in which exercises are sometimes expressed as when the pupil is told to find how many there are under certain conditions and the interrogative form as when the

pupil is asked how many there are after a situation has been described Distinctions of the type described and combinations of these different kinds of questions may properly be noted in studying textbooks On the other hand, one who is analyzing textbooks may find it more advantageous to classify together all cases of buying which call for addition whether the objects purchased be books or articles of food, one may classify together imperative and interrogative forms The analyses which are reported in the remaining pages of this chapter followed the plan of overlooking the finer distinctions

The analysis of the problems in addition in the four sets of arithmetic texts yielded a list of 410 types of problems which can be described as different without resorting to all the possible distinctions Furthermore, it was found that the kinds of problems which appear in one text are not always paralleled by similar problems in the other texts One author makes much use of farm problems, another uses space problems in greater numbers One author expresses the question by inserting a blank line to indicate that the pupil is to supply an answer, as, for example, "7+5 are ——? " Another never uses this device When the usages of all the authors are assembled in a single list, they make a formidable array of variations

In the present inquiry no effort was made to discover whether individual teachers exhibit personal preferences for certain forms of questions It is entirely thinkable that some of the confusion experienced by pupils in arithmetic may arise from the fact that teachers have favorite lines of concrete interest and special arithmetical vocabularies, as authors certainly do

TYPES OF PROBLEMS IN ADDITION

A general view of some of the major types of problems in addition may be useful as contributing to a fuller understanding of the complexity of the experiences encountered by pupils in studying arithmetic texts The following words calling for addition were found in the arithmetic texts analyzed

Add Find the sum Find the total Find the amount Find the value Find the cost Find the distance Find the whole Find the length What is the sum (total amount value cost, distance whole length)? How many? How many in both? How many together? How many in all? How far? How long? How old? How much are they? What is the combined cost (distance length)? What is the price?

Some of the foregoing formulas are not strictly arithmetical because they include references to the situations Thus, when the word ' cost is

used instead of such a word as "sum" or "total," an element of concreteness is introduced

When the situations which are involved in the problems are considered, the list is much longer The following items suggest the range of problems buying and selling, measuring space and time, earning and saving, finding, acquiring by cultivation, enumerating people, enumerating articles of furniture and other kinds of equipment, measuring in all the cases in which weights and measures such as pounds and quarts, are employed, scoring in games, recording the amount of rainfall or temperature, and totaling work or other accomplishments, such as the number of times one sees something In addition to this list, there is a long list of subtle problems which do not ask any very direct questions that give clues to the arithmetical procedures required The following are problems of this type ' It takes six hours to go from my house to my grandfather's house If I start at one o'clock, when will I reach my grandfather s? ' ' What is the average weight of five chickens which weigh 3 pounds and 10 ounces 4 pounds and 8 ounces ? ' In computing the average called for in the second problem, the pupil must first add and then divide In multiplication where multipliers of more than one digit are used addition is implied

CONTRAST BETWEEN PROBLEMS IN ADDITION AND CERTAIN OTHER PROBLEMS

The problems in subtraction are more sharply distinguished in their terminology from the problems in addition than are the problems in multiplication Indeed, it is very often quite impossible from the strictly arithmetical formulas employed in addition and multiplication to distinguish the two processes Thus a problem in addition will give the prices of several articles and end with the question ' How much did they cost all together? The corresponding problem in multiplication will state that somebody bought four articles at twenty cents each and will end with exactly the same arithmetical formula "How much did they cost all together? ' The clue to addition in the one case and to multiplication in the other is to be found in the earlier part of the problem where the conditions are stated In subtraction on the other hand, it is usually in the arithmetical formulas rather than in the description of the situation that the distinctive character of the problem is revealed Some of the most common expressions calling for subtraction are as follows

Subtract Take away Find the remainder Find the balance Find the difference Find what is left How many more? How much longer (taller heavier

farther)? How many are needed? How many are lacking? How many others are there? Not counting —— What are the rest? How many more?

SUBTRACTION AND ADDITION IN PRACTICE AND IN LOGIC

It is very striking that the words which are used in calling for subtraction and those which are used in calling for addition usually have no resemblance We are accustomed to recognize the fact that, logically and mathematically, addition and subtraction are closely related, one being the exact reverse of the other The verbal statements give no clue to this fundamental mathematical relationship There is in this fact very impressive evidence of the difference between the nature of the abstract number system with its laws of combination and the practical use of the number system in concrete situations where the character of the procedure adopted grows out of the concrete demands In practical life the process of subtraction is not a negative fact It is very emphatically a positive fact One must actually withdraw a certain number of objects It is only when the practical situations requiring addition and subtraction are reduced to numbers and are dealt with through manipulations of numbers that the mind discovers that the two procedures are fundamentally related that one is the inverse of the other In the world of number relations addition and subtraction are intimately related, in the world of practical affairs they are totally different and widely separated

The contrast here pointed out is of importance for the understanding of the methods employed in present day arithmetic texts One author has made the following comment in the Preface of the first book of his series

> It will be observed that in the early steps in subtraction the pupil learns to derive his facts about $8-5$ $6-2$ etc., from his knowledge that $5+3=8$ $2+4=6$ etc but that care is taken that he distinguishes subtraction sharply from addition gives it its proper name understands its common uses and soon comes to think of subtraction combinations fluently and directly This is important [1]

In the paragraph here quoted the arithmetical aspects of subtraction and the practical aspects are both clearly recognized Furthermore, the author is clear in his assertion that it is his judgment that in the earliest stages subtraction should be based on mathematical preparation and mastery of addition

A like position with regard to the method of teaching subtraction is taken by another author in the following statement

[1] Edward Lee Thorndike *The Thorndike Arithmetics* Book One p vi Chicago Rand McNally & Co 1924

Economy demands that subtraction be taught as the inverse of addition To teach the tables as *new* facts is inadvisable because there are 100 of the facts including those with zero When taught as the inverse of addition they are only new forms of facts already known That is knowing that 5 and 3 are 8 the child can answer the question 5 and what are 8? [1]

This second quotation differs from the first in that it does not emphasize the contrast between addition and subtraction This author is wholly engrossed in the mathematical and logical relationship He does not seem to be impressed by the fact that all the words used in problems in subtraction tend to obscure the strictly mathematical fact which he pushes into the foreground In spite of this devotion to the purely mathematical relation between addition and subtraction this author expresses as follows the first case of subtraction which he presents in his text "John wants a top that costs 9 cents He has 6 cents How much more does he need? The terms here employed do not repeat any which are used in the preceding pages of the text dealing with counting and addition The child is called on to recognize a fundamental mathematical similarity in spite of a striking difference in words This is a demand for a very high type of abstract thinking

The authors of a third text do not deal explicitly with subtraction they seem aware, however, of the difficulty of translating verbal problems into mathematical problems They say

The first difficulty that confronts a pupil in the second grade when he attempts to solve a concrete problem is the language difficulty Attention is called to the method used in this text to minimize this difficulty The concrete problem is stated in full for the teacher to read so that the pupil gets the concrete background Each problem is followed by a condensed statement of the problem which the pupil is to read and answer After hearing a large number of concrete problems read and abbreviated in this way the pupils learn how to select the process and condense the problem into the brief form [2]

Turning to the pages in which these authors introduce subtraction, we find the first exercise to be a strictly mathematical exercise 5 and ? = 9 If we take 5 away from 9 how many are left? The first concrete problem is as follows May, Ethel and Jane were making a seven room play house in the leaves When they had made 4 rooms how many had they left to make? 7 rooms—4 rooms = ? rooms

[1] John C Stone *The Stone Arithmetic* Primary p vi Chicago Benj H Sanborn & Co. 1925

[2] Charles E Chadsey and James H Smith *Efficiency Arithmetic* Primary pp iii iv New York Mentzer Bush & Co 1920

EARLY SOURCES OF CONFUSION IN ARITHMETIC

The foregoing quotations and the contrast drawn between the words used in referring to subtraction and the words used in referring to addition show that arithmetic is a very complicated subject Before they have developed the power of abstract thinking pupils in the earliest grades find themselves confronted with a very elaborate number system which has properties of its own They also find themselves required to use this number system in various ways in dealing with situations which are described in a bewildering variety of terms Not infrequently the difficulty of learning arithmetic is increased by the fact that neither the textbook nor the teacher gives any adequate explanation of the way in which the abstract ideas which the pupil must learn to understand are to be extracted from problems or of the way in which concrete situations are to be reduced to purely numerical formulas

NUMBER OF DIFFERENT PROBLEMS IN SUBTRACTION

The total number of different ways in which the idea of subtraction is expressed in the four sets of arithmetic texts analyzed is somewhat less than the total number of ways in which the demand for addition is expressed Three hundred and seventy four varieties of problems in subtraction were found The situation phases of the problems need not be described in full Buying and selling and journeys and measurements appear here quite as frequently as they appear in addition There are a number of problems of comparison which are not duplicated in addition The following are typical problems of this kind John is five feet tall Mary is six inches shorter John is 17 years old and Mary is 3 years younger

FUNDAMENTALLY DIFFERENT TYPES OF SUBTRACTION

While the number of ways in which subtraction is expressed is somewhat less than the number of ways in which addition is expressed the problems which appear in subtraction are far more varied in their fundamental character than are the problems in addition There is one broad general concept pervading all cases of addition that is the concept of assembling or uniting groups of objects In subtraction the notion of withdrawal or taking away is by no means universal The idea of comparison is quite as common The subtraction involved in comparison is but one step of a larger operation When John is older than William one finds the degree in which John is older by subtraction but one does not take away anything from John s age one takes away the number representing William s age from the number representing John s age in order

to compare the two ages The subtraction is only the first step in the larger process of comparison This subordination of subtraction to comparison makes the general concept of subtraction very much more complex than is the concept of addition

CHARACTER OF MULTIPLICATION

Multiplication is logically and mathematically a special case of addition In practice, also, multiplication and addition are closely related The close logical and practical relation between the two arithmetical operations is apparent from the way in which problems are expressed As was stated in an earlier paragraph, problems are very often so much alike in the two kinds of operations that the verbal formulas used in the questions are almost indistinguishable The characteristic element of a problem in multiplication is very often to be found in the description of the situation rather than in the arithmetical formula For example, when it is said that Mary bought seven yards of cloth at 39 cents a yard and the question is asked, "How much did the cloth cost?' there is nothing in the formal question to indicate the mathematical operation required The clue to the necessary operation is in the phrase 'a yard' This indicates that the cost of each yard of cloth is exactly the same Somewhere in every multiplication problem there is a word or phrase indicating that the same quantity is repeated Such words and phrases as 'times," 'per," "each," "on the average," 'a day,' 'a week a yard, "an acre," and "at the rate of" are characteristic of problems in multiplication Examples are as follows "How far is it three times around the lot?' "Costs 40 cents per yard" "Received $6 00 for each 'Saved on the average each week" 'Earned $2 00 a day 'Sold at the rate of '

Sometimes the idea of repetition is not made at all prominent, as, for example, in such cases as the following 'The second and third grade rooms each have 36 pupils Find how many there are in both grades" This is intended to be a problem in multiplication but in form it is very much like a problem in addition Another problem of this type is as follows "Bob receives 35c for working an hour If he works a full week of 48 hours, how much will he receive?'

There is great confusion in the minds of many pupils with regard to multiplication This is hardly to be wondered at when one looks at the arithmetic texts and notes the little definite help that is given the pupil in gaining a clear understanding of the operation The books usually introduce multiplication by giving exercises in counting or exercises in addition Thus, one book introduces examples requiring counting by five s

and then passes to problems in purchasing involving repeated five s The *next exercise demands counting by ten s Another book sets down the first* examples in multiplication as though they were examples in addition as follows

2	8	5
2	8	5

A third book does the same somewhat more systematically as follows

1	2	3	4
1	2	3	4

CONFUSION BECAUSE OF FAILURE TO DISTINGUISH OPERATIONS

The *pedagogical and psychological principle* to which these authors would undoubtedly appeal in defense of their method is the principle that in introducing a new procedure use should be made of all the earlier related experiences in the minds of the pupils It is evident however from the confusion which arises in the minds of pupils that too much emphasis on *similarities in procedure makes it difficult for pupils* to gain a clear conception of the special and distinguishing characteristics of a new operation The mathematical fact is that multiplication is a special kind of procedure depending on the repetition of a single quantity To express the matter otherwise the small groups which go to make up an aggregate product in *multiplication must* be of *identically* the same size This fact sharply distinguishes multiplication from most of the cases of addition The most common case of addition is that in which the groups which are being combined are not of the same size

DEPENDENCE OF MULTIPLICATION ON THE ARABIC NUMBER SYSTEM

Historically multiplication was not common and was hardly possible before the evolution of the Arabic number system The special character of this number system makes it possible to deal with the idea of repeated quantities very easily The same symbol is used for 5 units 5 *tens 5 hundreds and so on thus calling* attention to the fact that repetitions of a given quantity give results which have properties like the repeated quantity This fact was so completely obscured in the Roman system where different letters had to be employed for all the higher levels of number thinking that the Romans possessed the useful instrument of multiplication only in the most limited degree

MULTIPLICATION AND LARGE NUMBERS

Multiplication is one of the most useful ideas that arithmetic can give a child as a means of training him in the comprehension of large numbers. As has been shown in earlier sections of this chapter, addition is closely related to the development and use of large numbers, but the extent to which the mind can follow the results of addition is very limited. The numbers produced by addition soon pass beyond direct comprehension and are lost in the highly nebulous type of experience which always results when numbers become too large to be interpreted directly. Multiplication, on the other hand, can produce very large numbers in a way which is fairly comprehensible. Thus twenty-four 20's is a fairly easy combination to understand, while 480 is very vague and continues to be so even if it is thought of as made up of 190 and 290.

MULTIPLICATION AND LARGE ENTERPRISES

The modern use of number in the industries and in science can fairly be said to be characterized by emphasis on multiplication as distinguished from addition. Merchants do not estimate the value of their stock by adding items as a Roman merchant perforce had to do. A modern merchant makes his inventory very largely in terms of an enumeration which is turned into a statement of values by multiplication.

MULTIPLICATION IN THE TEXTBOOKS ANALYZED

The relatively large importance of multiplication is attested by the space allotted to this process in the textbooks analyzed. It is sometimes difficult to determine the amount of space given to multiplication because it is not easy to decide whether a problem is one in multiplication or one in addition. When there is more than one digit in the multiplier in an example in multiplication, there is always an element of addition involved in the process of securing the product. For the purposes of the analysis which was the basis of this chapter, any problem involving multiplication was classed as a problem in multiplication even when it involved also some other process, such as addition.

Five hundred and twenty-one kinds of problems in multiplication were found. The aggregate number of such problems is larger than the aggregate number of problems in either addition or subtraction.

The large number of different types of problems in multiplication will be readily understood when it is remembered that practically all the arithmetical cases and all the situation cases reported in addition are repeated here. For example, in such a problem as the following the formula

"find the cost of both" is identical with one of the formulas found in addition "Nell bought 2 boxes of candy at 39 cents a box Find the cost of both" In like fashion, such terms as "all," "the whole," and "all together" are common to addition and multiplication Besides the cases in which multiplication is like addition, there are special terms peculiar to multiplication Several of these have been mentioned, such as "times," "per," "on the average," and "each"

VARIETY IN INTERESTS OF AUTHORS

As one analyzes the different texts, it is very impressive to note the points at which different authors develop certain lines of thinking which are not followed at all by other authors For example, one author finds a fertile field in which to illustrate and expound multiplication in examples dealing with groups of people the number of boys on the basket ball team, the number of pupils in a room, and so on Another author deals with planting rows of seeds, another deals with cooking and dressmaking It may be remarked again that the variety in textbooks suggests the possibility that variety of an even more bewildering type may appear in classrooms where the personal preferences of the teachers are added to those of the textbook makers

DIFFICULTY OF DIVISION GENERALLY RECOGNIZED

Division has long been recognized by teachers as more difficult than any of the other fundamental processes Long division not only is difficult as division but involves enough multiplication and subtraction to make it a very complex process because of the different kinds of operations which must be performed in securing a result

COMPLEXITY OF THE MENTAL PROCESSES INVOLVED IN DIVISION

The reason why division is difficult can be explained psychologically by the fact that there are four distinct kinds of considerations to which attention must be given in every case of division while in each of the other fundamental processes only three such considerations or factors are involved When a person adds two quantities, attention must be given in succession to each of the quantities and to the result which is secured by combining them In subtraction, attention must be given to the two quantities which are present and to the result which is secured by comparing them In multiplication, there are the multiplicand, the multiplier, and the product, in this case the transition to the product is complicated by the fact that in reality, the multiplicand is not a single quantity but a repeated quantity If one thinks of the process of multiplication in all

detail therefore one must recognize the multiplicand as a complex factor or as a collection of factors The complexity of multiplication is evaded in most cases by substituting a remembered rule or a remembered product for the full process When one multiplies 7 by 3 one does not actually think of 7 and 7 and 7 one merely recalls a table which states that 7 taken 3 times equals 21

In division the complexity which is avoided in multiplication by remembering a rule is only partly overcome by learning a table The dividend and the divisor which are the first subjects of attention in division are always present The quotient is also a factor but one cannot think of it in its final form until one has thought of it in terms of a number of smaller equal groups into which the dividend is to be broken up One must consciously perform a trial multiplication thus interposing in division a step which has no corresponding parallel in the other fundamental operations or one must explicitly think of the smaller like groups which result from division or finally one must recall a rule or a table with the consciousness that the rule or table is a derived fact involving multiplication

The foregoing statement can be made concrete by means of an example ' Mary has 40 roses with which she intends to make 8 bouquets How many roses can she put in each bouquet? The process here involved is the discovery of the fact that there are eight small groups of five roses each which can be secured by subdividing 40 One can substitute for thought about the smaller groups trial multiplication with 8 as a multiplicand and various numbers tentatively adopted as multipliers otherwise there is no escape from some kind of consideration of eight groups If the meaning is recognized at all the number 5 in the quotient is not thought of as closely related to 40 It is a subdivision of 40 and its character is determined not so much by the 40 as by the requirement that there must be a partition into eight subdivisions The same 40 could be divided in other ways and both the number and the size of the resulting groups would be affected This statement is true even in the final stage of division where the mental process is dependent like much of the multiplication used by trained adults on memory of tables

MEANING OF THE VARIOUS FUNDAMENTAL OPERATIONS

The importance of the distinction which has been drawn between division and the other fundamental processes can perhaps be emphasized by asking the question What do the various results of the different fundamental processes mean? This question may be answered in the case of addition by saying that the result can be thought of as including all the

items dealt with in the example Any result secured by addition is a tangible outcome which can be thought of without reference to the process by which it was derived When one has put together five things and seven things, one has a large group of twelve things The process and its result are complete In subtraction of the "take away" type, similar final results are secured In subtraction of the 'comparison type, as when one asks how much taller John is than Mary, the result is in many cases a fact which can be understood only by thinking of all the items originally given In this case, however, though one must go back of the result to understand it fully, one does not have to think of anything except what was actually present at the beginning of the process of subtraction Multiplication is like addition The result is a fixed fact, relatively independent of its mode of derivation When 7 is multiplied by 5 one can think of the product 35 as a definite outcome of the process In contrast with all these cases, the result secured by dividing is always a partial outcome If roses are divided into bouquets one has to make the transition in thought from one small group to another Any single number which results from division can never be thought of as final in itself It is always part of a system of experience If one tries to understand a quotient, one has to think of several items of experience in order to include all that is necessary for interpretation Furthermore, the essential outcome of division is not in the things divided but in their arrangement The fact of arrangement is much less tangible and final than is the outcome of the other fundamental processes

DIVISION IN THE ARITHMETIC TEXTS ANALYZED

The conclusion which has been reached with regard to the complexity of division is fully supported by an analysis of the problems in division in the four series of arithmetic textbooks In the first place, there are a great many ways of expressing the requirement that a group be divided Questions and phrases such as the following are used 'How many will each have?' 'How much will it cost each member of the party? 'How many are used per day?' 'How much must be used on an average?' 'How many can he buy with two dollars at the price of 25 cents each? 'How many rows will a group of 35 pupils make when there are 7 in a row?" These questions are somewhat more complicated than are the corresponding questions found in the problems requiring addition subtraction, and multiplication In the second place division situations are complicated by the fact that sometimes the question asked shows that the pupil must keep in mind a partly performed operation

This is true of all fractional expressions John is giving his brother one-half of what he has, or Mr Brown is selling at a profit of one-quarter of the original cost In such cases the division required is expressed in terms of what may be called a "standard type" of incomplete numerical partition The fractions ½ and ¼ are conventional cases of division which are applicable to all kinds of situations and are so frequently employed that they become almost as fixed in thought as the number names

FRACTIONS

Fractions are sometimes defined as divisions of a unit The fact is, however, that the most common use of fraction names is to indicate a desired division of some quantity which is not a single unit One half a pound and one half a yard are quantities that are in reality not partial in any proper sense so far as the commodities of which they are composed are concerned One half a yard of cloth is a very substantial fact in the world When it is thought of as a factor in actual life, it is not part of something It is not deficient in any of the qualities of cloth It is less in amount than certain other quantities but the fractional character of the description is not reflected in any actual fractional appearance of the cloth

In the textbooks analyzed there are numerous examples of the use of fractions as means of inducing pupils to perform the operation of division The following problems are typical 'How many minutes in ¼ of an hour? 'What is the cost of ¼ of a pound of tea which costs 60c per pound? 'We used ½ of a can of paint which contained ⅞ of a gallon How much did we use? Janet is 9 years old her baby brother is ⅓ as old How old is the brother?

LARGE NUMBER OF PROBLEMS IN DIVISION

The use of fractional expressions enlarges very greatly the range of arithmetical formulas found in the textbooks analyzed which must be classified under division A total of 594 kinds of problems in division was found in the arithmetic textbooks analyzed This is the largest number of kinds of problems found for any of the fundamental processes

DIFFERENT USAGES BY DIFFERENT AUTHORS

The figures which have been presented in this chapter with regard to the number of different kinds of problems used in the textbooks analyzed serve to make it very clear that pupils face a bewildering array of different expressions in the study of arithmetic Even if we correct the extreme

impression made by the figures reported by recognizing the fact that each individual author uses only part of the forms enumerated there is still evidence that pupils are confronted by a serious intellectual task when they try to understand all the cases which fall under the four fundamental operations Of the 410 types of problems in addition Alexander Sarratt uses 141 Chadsey Smith 169 Stone 75 and Thorndike 1?6

RESULTS OF GROUPING LIKE CASES

A second partial reduction of the figures can be made by recognizing that certain closely related types of situations are capable of being grouped together For example the original enumeration distinguished the cases in which a child is described as earning money from the cases in which a child had money given to him In each of these groups of cases addition or multiplication is required that is they are cases of positive accumulation of money as distinguished from negative cases such as those involving expenditure or loss

Before one groups together cases in which money is earned and cases in which money is otherwise acquired one should clearly recognize the fact that a mature abstract idea of positive accumulation is implied as existing in the mind of the reader of the problems The word earned and the phrase received as a gift are not identical and do not belong together except in the thinking of an individual who has converted the two distinct concrete ideas into abstract arithmetical ideas When pupils begin to study number relations they do not have the general idea of addition and they cannot bring together in their thinking earning and receiving as a gift except as they cultivate the more general notion of accumulation

The classification of cases under headings that seem very proper to an adult may therefore be very arbitrary from the point of view of an immature pupil The only justification for attempting to reduce the number of categories used in analyzing the material in the textbooks is the desire to avoid all appearance of exaggeration of the difficulty encountered by pupils.

The attempt was made to reduce the categories of classification to the smallest legitimate number and to classify problems in groups by putting together problems which are very much alike although not identical either in concrete circumstances or in arithmetical formulas The 410 types of problems in addition were thus reduced to an absolute minimum of 260 types A number of experienced teachers to whom the

classifications were exhibited favored the number 410 rather than the reduced number as representing more adequately the demands made on pupils

NEED OF ANALYSIS

This being true it behooves all who are concerned with the teaching of arithmetic to undertake anew an analysis of the difficulties which pupils encounter in arithmetic to seek by every means available a better understanding of the reasons why pupils fail and to devise, if possible better methods of introducing pupils to the numerous and intricate ideas with which arithmetic deals

NECESSARY ANALYSIS IS PSYCHOLOGICAL

The contribution of this chapter to the psychology of number is a body of evidence which shows that pupils are asked in modern courses in arithmetic to cope with an enormously complex body of material and that they are given relatively little help in discovering the character of the underlying logical principles on which the arithmetical operations depend

Even in grouping concrete situations current textbooks are very inefficient There are levels of abstraction which must be reached before pupils can classify together those situations which imply positive combinations as distinguished from negative withdrawals The first levels of abstraction require cultivation of powers of mind which pupils do not exhibit when they first come to school The various levels of abstraction can be reached only by carefully guided mental efforts Modern textbooks too often gather together elaborate collections of problems but give no real help in reaching the levels of abstract thought necessary for their solution

Our analyses lead us to a recognition of several levels of thinking The lowest level is the recognition of a concrete situation in which something is to be done with a group of things The way in which the group is to be manipulated must be understood Often the group is manipulated most readily by being translated into an equivalent group of tallies There is a great variety of ways in which the rearrangements called for in arithmetic problems are to be made Certain of these rearrangements are generically alike and they may be classified together as additions Others are generically alike in a totally different way and must be classified as subtractions Addition is a word which can be used to describe a redistribution of concrete objects or a redistribution of tallies The same can be said of subtraction multiplication and division

Early in their school lives pupils are called on to master as best they can this complex of more or less similar concrete situations and more or less systematic groupings of tallies It is little wonder that many of them fail to understand the processes of arithmetical combination One of the chief reasons for the failure of pupils in arithmetic is that general ideas are not explicitly taught Teachers expect pupils to think abstractly and to generalize when they have had no adequate instruction in these difficult intellectual processes To correct the present unsatisfactory condition arithmetic must be understood with respect to its psychological complexities as well as with respect to its mathematical characteristics

CHAPTER VI

OUTLINES OF A PSYCHOLOGY OF THE FUNDAMENTALS OF ARITHMETIC

PURPOSE OF CHAPTER

This chapter has a double purpose It is intended to serve as a summary of the investigations reported in the preceding pages It is at the same time an attempt to formulate a psychology of the fundamentals of arithmetic in such a way that it can be accepted as a basis for methods of teaching

METHODS OF THE INVESTIGATIONS

Three methods of inquiry were employed in the investigations reported A very large part of the material for this monograph was collected by experimental methods Apparatus of a type familiar in psychological and physiological laboratories was drawn into the service of education and analyses were made of the mental processes of children and adults by means of measurements and comparisons of their reactions and introspections under rigidly controlled laboratory conditions A second method employed was the systematic analysis of the materials used in classroom teaching Arithmetic as taught in the schools is the outcome of a great deal of practical experimentation on the part of intelligent teachers The methods which they have adopted in presenting number facts to pupils are not mere accidental procedures They are the results of gradual adjustments and readjustments which have been made in view of difficulties and successes observed in classrooms What is now embodied in textbooks is therefore legitimate material on which to base inferences with regard to the processes by which pupils acquire number ideas A third method, which is not as fully illustrated in this monograph as are the other two methods, is the historical method In the first chapter a brief summary was presented of some of the leading facts with regard to the racial development of number systems The present writer has elsewhere[1] brought together more material than is here presented with regard to racial evolution in this field Enough has been presented in the various discussions of this monograph to indicate that the number system is a product of social co-operation and not a matter of individual invention

[1] Charles Hubbard Judd *Psychology of Social Institutions* chaps v viii New York Macmillan Co, 1926

STANDARD TESTS

One method which of late has been widely used in the study of arithmetic is referred to only incidentally at various points in this monograph, that is, the method of testing pupils' achievements by the use of standard tests. No extended applications of this method are reported in the foregoing chapters. Two reasons can be assigned for neglecting this common method of collecting information about arithmetic. In the first place, much, if not all, of that which can be learned by the use of tests has been very fully ascertained and reported by careful and competent workers. In the second place, the test method does not contribute greatly to the detailed analysis of mental processes. Tests are excellent devices for comparative studies of the survey type. They were formulated especially for that purpose. They can also be advantageously used in locating the difficulties and deficiencies of pupils. Where analyses of the processes of thinking are to be made, tests prove to be inadequate.

LIMITATIONS OF THE PRESENT INVESTIGATION

By the adoption of experimental, analytical and historical methods, it has been possible to bring together a large body of facts regarding the fundamentals of arithmetic. The limitations of the studies reported are many and fairly evident. Perhaps it will be well to make the explicit statement that this monograph does not lay any claim to being a complete treatise on the psychology of number consciousness. It does not touch on such important matters as the psychology of decimals and the psychology of involution and evolution. It does not deal at all exhaustively with denominate numbers or with the psychology of measurement. The field of this monograph is the field of early instruction in arithmetic. The period during which pupils are becoming acquainted with the fundamental ideas and processes of rudimentary mathematics is the period covered in the series of investigations reported.

NUMBER NOT A PRODUCT OF INSTINCTIVE REACTION

Certain general conclusions are justified by the facts discovered and described. The first of these can perhaps be stated most forcibly by using negative terms and by rejecting a statement which is sometimes made in educational psychology. Arithmetic is not a product of instinctive interests or tendencies. The individual does not develop an interest in number because of any native or inherited disposition which impels him to be exact in his thinking. In fact, all the available evidence goes to show that number is a very late product of social co-operation. Men did not count

and calculate until they had reached a stage of life where property had accumulated and until intercourse between members of the social group had reached the point where exactness in thought and expression was required by the group and was imposed on the individual by the group

PRIMITIVE PERCEPTION OF QUANTITY

There is doubtless present in all consciousness some evaluation of quantities One recognizes the familiar objects of furniture in one s room and the members of one's family before one can count Recognition of the primitive type here described is probably found in animals as well as in man There is, however, a vast difference between being able, on the one hand, to recognize familiar groups and being able, on the other hand, to perform the elaborate mental act of explicitly defining the number in such a group

DISCRIMINATION AN ACTIVE PROCESS

The steps by which the first recognition of groups is converted into an explicit recognition of number are as follows The observer must first discriminate the individual items which make up the group Discrimination is an active process it may be described as dependent on reactions in which the observer gives individual attention to each of a succession of items and makes a distinct movement corresponding to each item The difference between the successive movements is often not great A gesture of pointing is an individual reaction A second gesture may be in almost the same direction as the first but if there is a recognizable difference in the direction of the gesture it will serve to mark the two objects of attention as distinct from each other in location The two gestures which are here described as means of discrimination may be extended so as to become characteristically different in respects other than spatial direction Primitive peoples use elaborate methods of pointing in helping themselves to discriminate objects They point first with the hand stretched out, then with one finger bent then with two fingers bent and so on The characteristic gestures thus developed are both aids to discrimination and means of retaining in mind the series as a whole

The motives which a person may have for separate reactions to the members of a group of objects are often various and do not in all cases lead to counting In practical life some objects are discriminated because they are grasped some objects are discriminated because they are rejected Active discrimination in such cases is seen to be much broader in its scope as a form of human behavior and human experience than is number consciousness One of the most important generalizations of

modern psychology is the principle that all attention and all discrimination are conditioned by reactions of some type

NUMBER AS A LATE PRODUCT OF ABSTRACT THINKING

The intellectual step which must be taken in passing from the simplest and most immediate discriminative reactions to enumeration is a long one and involves as antecedent steps the evolution of language and the evolution to some extent of the power of abstract thinking That language in general precedes the production of number names in particular is shown by the fact that semi barbarous tribes which have very few number words have relatively large vocabularies of names for the objects which they possess Not only do they have words, but they are able to use these words as designations of whole classes of objects Words help in the formation of class ideas and in this way promote abstract thinking The foregoing statements are confirmed by the fact that the most primitive forms of enumeration consist in calling off the names of the objects which are being enumerated Even now, when we are in full possession of a series of number names, we designate the members of the family by names, and whenever we have occasion to find out whether all are present we use names, not numbers In like fashion we sometimes call the roll of a school class or of a company of soldiers by using names These are cases in which the interest of the enumerator in the individual members of the group is strong

NUMBER DEPENDS ON INTEREST IN OBJECTS OF UNIFORM TYPE

One has to think of the transition from enumeration by naming to enumeration by counting as taking place when men became interested in groups of objects—probably possessions—which were so uniform in quality and in appearance that interest in the individual object receded and interest in the magnitude of the group became prominent

SOCIAL DEMAND FOR EXACTNESS

There is also probably to be postulated as necessary to the evolution of a number system an element of social competition or social demand for exact communication Men began to exchange commodities with one another and, in so doing demanded of themselves and of those who entered into barter with them exactness of statement about the items involved

TALLY SYSTEMS

As necessary devices for counting and as means of serving the interests of the social group tally systems were evolved The wampum of the

Indians, the pebbles of the Romans, other objective facts such as sticks or marks and, above all, the fingers were called upon to help the minds of primitive men in being exact Men used these tallies when they described to themselves and to those with whom they had dealings the magnitude and value of groups of objects

SOCIAL CHARACTER OF THE IDEA OF ACCURACY

It is to be noted that there came into the life of man through the evolution which is being sketched a new kind of demand, born of social relationships, namely, the demand for precision and accuracy It used to be said with complete assurance that arithmetic cultivates certain general mental attitudes Of late, it has become the custom to deny all general training because it is assumed that belief in general training involves acceptance of the doctrine of formal discipline in its most extreme form The evidence derived from the experiments reported in this monograph shows that school instruction may produce general ideas of addition and subtraction and, above all, general ideas of the systematic character of the Arabic number series Indeed, we can go much farther in refuting the position which has frequently been taken by those who deny general training We can assert with a high degree of certainty that the result of all good teaching is the cultivation in the minds of pupils of broad general ideas which could not have arisen through the simplest type of perceptual recognition of objects As soon as the mind of man developed names for classes of objects generalized thinking began As soon as values began to be recognized in various objects of the environment, man developed the general idea of worth and began to demand the description of this quality General ideas represent the mature products of experience when experience has accumulated to the point where it is something more than mere recognition of objects

EVOLUTION OF THE GENERAL ATTITUDE OF EXACTNESS

Among the most important general ideas which the race has developed is the idea of exactness Children do not have this idea any more than savages in the first stages of social life have it In the individual, as in the race, experience has to mature and social demands have to drive the individual to the consideration of something which would not have emerged as a clear idea if there had not been social pressure for its recognition

A PERFECTED NUMBER SCHEME HISTORICALLY LATE

The evolution of number in the history of the race was very slow Crude methods of expressing number continued to be the only instru

ments of trade and measurement even in historical times The appearance of a highly perfected system capable of expressing manifold number relations and capable of being used in intricate processes of calculation is a matter of comparatively recent history The Arabic number system came into Europe and superseded the clumsy Roman system in the Sixteenth Century

CHILDREN'S DISCRIMINATION GRADUALLY DOMINATED BY SOCIAL MOTIVES

What happened in the racial evolution of number is repeated in some of its more prominent phases in the life of the individual child The child begins life without clear discrimination of the objects of his environment As the child's experience progresses certain objects are selected for specific reaction and for discriminative attention Discrimination is, at first a result of the maturing of the personal interests of the child There is a positive response to those objects which afford comfort and there is a specific reaction of rejection to those objects which are found to be disagreeable Later discrimination is guided by social rather than personal considerations Some of the objects of a child's environment are vigorously brought to his attention by his elders People themselves are so conspicuous a part of his environment and such regular sources of comfort that the child learns early in life to distinguish people from things and bestows a very large share of his attention on people and their doings Attention is here, as in all other cases a matter of active response The child moves his eyes in following people about the room he smiles in response to social example he kicks and moves his hands and arms in delight when he is stimulated by the approach of some familiar person From this point on the development of the child's reactions and of his interests takes a direction which is determined in very large measure by his interest in people He is guided by those about him He becomes part of a social group his interests are those which the group as a whole emphasizes

CHILDREN SUPPLIED WITH SCHEMES OF REACTION

The child in a modern civilized family finds the group interests and group reactions very highly evolved and fixed by long usage If we think, for example of the complex reactions which make up language, we realize how elaborate are the systems of behavior of a civilized social group In support of his own interests and in response to the coercions of society, the child is compelled to cultivate as rapidly and as fully as he can language reactions which correspond to the accepted conventions of the

group which surrounds him His task in this case is not to invent language but to adjust himself to the social scheme

NUMBER AS A SOCIAL SCHEME IMPOSED ON THE INDIVIDUAL

What is true of language is emphatically true of number The number system is a highly evolved system of social reactions which the child finds in the world into which he comes The child s relation to the number scheme is in many respects different from that of the race as a whole The race has passed through a long process of evolution in the course of which it has produced by exercise of its inventive genius various systems of tallies that have served to record its gradually increasing expertness in noting quantitative facts The race not only has evolved systems of tallies but has devoted much attention to the refinement of the system itself In the course of its experimentation, it has also evolved certain broad general ideas which result from the use of the number scheme Among these general ideas is that of exactness

The child comes into a social environment which is possessed of a highly perfected number system but the child feels no need of it and he is incapable of the abstract thinking that is necessary for the intelligent use of this system To the child the number system is in itself a body of very complicated experience He is forced to acquire the system in order to adjust himself to social demands He finds his social environment aggressively forcing on his attention a system of names and a mode of thinking which are not at all natural to him and not of his choosing

When one thinks of such matters as compulsory education and the school curriculum one realizes how aggressive society is in forcing the pupil to adopt its methods of intellectual operation Adults know that it will be advantageous for children to learn how to think with precision Adults know that the best system to use in order to think with precision is the system of Arabic numerals Therefore adults assemble children in classrooms and by every possible means strive to induce and compel them to adopt the Arabic numerals and all that these imply The deliberate effort of society is to give pupils in a few years and in its most highly perfected form that for which the race strove during long centuries

THE STEPS NECESSARY IN TEACHING AS CONTRASTED WITH THE STAGES OF HISTORICAL EVOLUTION

The number system to which the race introduces the pupil through school instruction must be resolved for purposes of school use into a number of simple phases The system cannot be taught without being subdivided because it involves a variety of different types of mental activity

for its complete mastery, furthermore, it involves various levels of abstract thinking, the highest of which can be attained only when the lower have been fully mastered The analysis of the number system for purposes of instruction will not necessarily result in the same internal divisions of arithmetic as those which are suggested by the history of racial evolution For example, there was a long period when all the combinations, even addition, were made with the aid of material objects, which served as tallies Modern civilization is in possession of a number system so highly perfected that mechanical aids in computation can be dispensed with Another example of the modern perfection of the number system is to be seen in the fact that the nine digits and zero can be used to express all the possible number ideas The older method of using different symbols for different numbers is still reflected in the number names where the words 'hundred,' "thousand' and "million' mark stages in counting which are thought of as new For purposes of calculation, the race has reduced to a simple system of digits all the differences which these names express When pupils learn the rules of calculation, they can take advantage of the system of digits which shows that 10 and 100 and 1,000 are alike in their fundamental properties in spite of the marked differences in their names

TEACHING AS A PROCESS OF ADAPTING PUPILS THINKING AND THE NUMBER SYSTEM TO EACH OTHER

The conclusion to which our discussion brings us is that the child in a present day school is urged and even compelled by his elders to learn rapidly and without regard for his native interests a system of number and the corresponding abstract idea of exactness The devices which the child acquires are different in certain important respects from those which the race employed in the early stages of its experimentation with the system Though it is highly perfected the number system is difficult for the child to acquire because it is complex and the child s powers of thought are limited The procedure of guiding the child to the complete understanding of number will be successful only when there is an intelligent analysis of the number system on the one hand and on the other hand an equally intelligent consideration of the child s modes of thinking and possibilities of development in the mastery of abstractions

LAWS OF ASSOCIATION IN LEARNING NUMBER NAMES

It may seem to some readers to be superfluous to reiterate the fact that the process by which arithmetic is learned is one of two sided adapta

tion There is, however, much evidence to show that due consideration has not been given in the schools to the fact that the child s methods of thought must be understood if the teaching of arithmetic is to be a success For example the historical development of number undoubtedly de pended at every step on practical applications Names for the large num bers were not devised until they were required for practical purposes At the present time since all the number names are in the possession of society, it is more economical to teach children a series of names which is much more elaborate than practical needs dictate By its own nature, the mind tends to follow readily one line of association at a time The line of association which is established when the pupil learns to count is rela tively easy to carry forward when once it is started It would not be economical to teach pupils the number names "one' two ' and ' three and then to pause as did racial evolution and develop some of the rules of combination before going on to the number names four,' 'five,' "six,' etc

INDIVIDUAL METHODS OF MASTERING NUMBER

The psychology of arithmetic teaching is a mixture of the psychology of the number system and the psychology of the learner The number system is the embodiment and product of a long line of mental experi ments The number system has characteristics which result from the operations of the minds which produced it The number system has mature intellectual characteristics which can be described and under stood It is equally true that each human being who learns the number system and uses it has a number consciousness of his own This personal number consciousness is to be explained in part as imitating society s number system and in part as made up of the individual s own methods of thinking If the individual s mastery of the system is quite inadequate it often becomes necessary for society to condemn the results of number thinking which the individual reaches by his private intellectual processes For example if an individual believes that 7 added to 8 produces 17 society will discipline him If on the other hand the individual has cer tain private ways of thinking about number and using number which do not adversely affect the practical outcomes of thinking society is tolerant and will not attempt to coerce the individual For example certain per sons have what Galton described as number forms The number series is always pictured in the minds of such persons as arranged in a fixed spatial order This is a matter of purely private imagery The essential character of number ideas is not influenced in any way by this method of thinking of individual digits The number system has been mastered by

the mind in its own way, and society is not concerned to change the situation

RELATION OF EDUCATIONAL PSYCHOLOGY TO SOCIAL PSYCHOLOGY AND TO INDIVIDUAL PSYCHOLOGY

The double analysis of the psychology of arithmetic into the psychology of the number system on the one hand and into the psychology of the learning mind on the other is the duty of that division of science which is known as 'educational psychology' Educational psychology differs from social psychology because the latter deals not only with the perfected number system but also with the series of processes which begin where there was no number system Social psychology must trace intellectual progress from its historical beginnings to the point where a system has been evolved which is one of the most perfect achievements of group co operation As contrasted with social psychology, educational psychology is concerned primarily with the completed number system and its implications Educational psychology differs also from individual psychology The latter commonly overlooks the fact that there are two kinds of sources from which experience is derived the one source is physical objects which are important chiefly because of the sensations which they arouse in the individual mind the other source is psychical objects such as number language customs and government which are important not so much for the sensations which they yield as for the modes of arranging ideas which they induce in the individual s experience

NUMBER AS CONTENT AND AS METHOD OF THINKING

When educational psychology treats of number in the way which is implied in the foregoing paragraphs it will not describe number merely or primarily as a new content of experience It will describe number as a device for a highly evolved form of thinking This statement does not deny that the number system is content The learner must know one and two and the other number names and must know the digits which correspond to the number names He must also know the rules of various combinations and their results In these respects number is content of experience The value of number however is not in its contribution to the content of consciousness Number is a device for arranging many different facts of experience into groups and for combining and recombining these different facts of experience Number is thus seen to be a system of thinking Its place in the educational scheme is not primarily that of an informational subject It is a subject which teaches the individual to organize ideas in systematic and exact fashion

EXTREME PEDAGOGICAL ATTITUDES TOWARD ARITHMETIC

Such distinctions as have been made in the foregoing paragraphs can perhaps be shown to be of practical importance to the teacher by drawing a contrast between two extreme attitudes which are often taken in school with regard to arithmetic There is, on the one hand, an attitude that arithmetic is an interesting system of symbols and rules to be learned through vigorous drill and to be kept in the mind as a body of content There is on the other hand, the attitude that number should be taught only in so far as it is absolutely necessary in solving the practical problems of ordinary life Those who want to reduce the use of number as a school subject to its lowest possible terms hold that there should be a minimum of drill and a minimum of attention to the rules of number combination

CORRECT TEACHING RECOGNIZES NUMBER AS CONTENT AND AS METHOD OF THOUGHT

Somewhere between these two extremes lies the true psychological position The number system which the race has evolved is a complex of symbols and of rules of combination Some mental effort is necessary for the mastery of the system itself In so far as this is true arithmetic is a content subject Equally true is the statement that the number system is a means of arranging the facts of experience in such a way that they can be dealt with precisely although they are quite chaotic in their own quality and order of presentation Because the number system helps the individual to arrange his experiences it is the indispensable instrument of all science and of commerce where facts must be dealt with not in a chaotic way but in such a way that relations are definite and clearly recorded

Educational psychology of the type described in the preceding paragraphs teaches that when the pupil acquires the number series he comes into possession of a valuable guide to thinking he gains knowledge of an orderly arrangement to which he can match the miscellaneous experiences of life The fixed order which in earlier chapters was called the subjective counting series or the series of number names is acquired by the child with the other words of the vernacular as a part of the social system long before there is any urgency in the child s mind for precise and orderly thinking Indeed, the notions of precision and order are themselves products of drill and use not parts of the number system itself

NUMBER AS ONE OF THE HIGHER MENTAL ACQUISITIONS

As was stated in an earlier section of this chapter, number is not a product of instinctive reaction it is not inherited and it is not a matter

which children seek of their own initiative The expectation that little children are going to show a spontaneous interest in number such as they show in good things to eat or in bright colors or noises which appeal to their senses is certain to meet with disappointment The contrast between instinctive interests and interest in number is attested by the fact that number is not acquired by mental defectives One of the clearest marks of intellectual deficiency is lack of ability to understand even the simplest number expressions A mentally deficient child often has instinctive reactions highly developed but is without interest in number He is also without social interests and capacities Even when he responds to colors and to other sensory qualities, he does not possess any subjective systems, such as number

IMPORTANCE OF THE GROUP IDEA

The normal child acquires number by social imitation He first learns the number names from those about him These he matches one by one with various kinds of objects and gradually achieves the notion that a group of objects is a new kind of reality A group of five things is a reality distinct from the individual objects which compose the group Five people are not merely five separate experiences they are a group The group has a kind of reality which is broader and more inclusive than the individuals of which it is composed

METHODS OF COMBINING GROUPS

As soon as the pupil is able to recognize even in a primitive way the fact that numbers help him to group the objects of the world in which he lives a new type of thinking is possible He begins to think of combining and recombining groups The simplest way to change a group through a process of combination is to extend it Thus if one counts people and arrives at the point where six persons have been clearly distinguished the next step is to add a seventh At the stage where thinking deals with groups counting is something more than matching a subjective series with an objective series Counting passes into a higher stage and becomes addition

ADDITION MORE THAN COUNTING

It is important at this point that a psychological principle of the highest significance be explicitly introduced and understood The statement that addition is derived from counting and grouping experiences does not provide a full psychological account of addition The psychological description of addition is not complete until it is noted that the

mind which makes the transition from counting to addition has achieved a new and unique form of thinking which is more highly organized than counting and consequently of a higher type

HIGHER TYPES OF CONSCIOUSNESS PRODUCTS OF MENTAL DEVELOPMENT

The characteristic fact in all progressive mental life is the gradual achievement of new levels of experience through organization When the elements of experience are put together in complex organizations, new kinds of thinking are produced We express this by saying that the idea of a group is an abstract idea It is not like the succession of concrete ideas secured by observing individual objects An abstract idea is a comprehensive idea including individuals as distinguishable items and at the same time subordinating the individuals to the larger whole in which they are included

As soon as the mind reaches one superior level of thinking through the organization of experience, it is capable of taking the next step in advance Counting being matured into grouping it is now possible to deal in various ways with groups Thus groups may be compared with one another if they are not too complicated Groups may be reassembled or subdivided The mental processes which are evolved when groups are recombined in various ways are familiar and have been amply illustrated in earlier chapters

GENERAL IDEAS MATURE SLOWLY

From what has been said it will be readily seen why time and much training are necessary in order to mature the child s development of arithmetical ideas General ideas of what a group is and of the way in which groups can be rearranged come only through contact with many different groups When once achieved the higher type of thinking is continually enlarged and refined by being employed in different connections The idea of addition for example is not a fixed notion It cannot be given to a pupil It must be acquired through a long period of enlargement of experience in the course of which various kinds and degrees of recombination of groups are experienced

A very false and misleading psychological doctrine has been current in some quarters It is said that in the pupil s mind, addition is nothing but a long list of particular combinations Such psychology does not provide any explanation of the general or abstract idea of addition as distinguished from particular experiences When a teacher thinks of addi

tion merely as a series of separate processes he makes no provision in his teaching scheme for the operation of the generalized idea When, on the other hand, a teacher thinks of the idea of addition as a generalized idea, he will strive to emphasize this idea He will have confidence that, if the pupil is carried far enough to acquire an insight into the general idea, the pupil will, because of the inner drive of this general idea, help to train himself in particular cases not explicitly included in the lessons taught in the school When once acquired, a general idea is a dynamic fact in experience

GENERAL IDEAS HAVE REFLEX INFLUENCE ON THINKING

The second truth which is to be thought of in this connection is that, when a general idea is formulated, it reflects back on the experiences from which it was derived and gives to these experiences a new meaning and value which they could not have in a mind which has not developed the general idea The simplest numbers have a new value for the child who has learned to add Such a child will never use number names again without importing into them the feeling of possible grouping and regrouping It was shown in an earlier chapter that there is progress in counting throughout the grades even though there is no explicit instruction in counting during this period The improvement noted is part of the gradually expanding comprehension of number and of the gradually increasing familiarity with the modes of thinking that are involved in all abstract arithmetical thinking

ORGANIZATION THE MAJOR FACT IN MENTAL DEVELOPMENT

The principles which have been enunciated in the foregoing paragraphs can be summed up in the general statement that the character of every mental process is determined by its organization, not by its isolated elements

It has recently been contended by the school of so-called *Gestalt* psychologists that there are no elements in consciousness It is undoubtedly true that there are no isolated elements there are no elements which can be detached from the mind s mode of uniting them into patterns of experience For the student of educational psychology this means that number experiences are gradually matured through organization into higher and more general types of thinking Group ideas are of a higher order than the process of counting addition and subtraction are of a higher order than mere grouping With each step upward to higher levels of thought, the whole number scheme is seen by the learner in a new perspective, and each item of experience takes on a new value and meaning

FEELINGS OF CONFIDENCE

When number consciousness reaches one of the higher levels, such as are here under consideration, it becomes possible to withdraw attention from the simpler phases of experience For example, when one learns to add, one does not need each time to go through the full process of giving attention to all the individual items in the two groups which are combined The groups are assumed to be made up of items, and all that is necessary is to consider the groups as wholes The trained individual knows that he can, if necessary, go back to the lower levels of experience, and he sometimes does go back for purposes of verification Under ordinary conditions, however, he does not go back to the steps necessary for complete verification Consciousness at the mature stage includes a factor which represents to the mind the possibility of going through the longer process, this factor we may call a "feeling of confidence" The person who knows that 6 added to 7 equals 13 has, in addition to a knowledge of the sum 13, a mature attitude of mind which issues from all the earlier experiences with the two constituent groups and with the process of addition, this mature attitude of mind is a knowledge of the result of adding 6 and 7 combined with a feeling of confidence that 13 is the correct sum We describe the situation by saying that the experience of addition in this case is somewhat indirect, it does not have the immediacy of contact with its concrete background which the simpler process of counting has, and yet it is accompanied by a vivid feeling of confidence, which is the result of earlier training

HAZARDS OF INDIRECT TREATMENT OF NUMBER RELATIONS

When addition gets away from its complete dependence on counting, its indirect character introduces certain hazards into mental life which are well known to all teachers of arithmetic It often happens that pupils attach the attitude of confidence to results which do not merit confidence In short, wherever a mental process becomes detached in any degree from its direct dependence on concrete experience, the possibilities are opened for a variety of interpolations of foreign and undesirable elements which endanger the validity of the process

KNOWLEDGE OF THE USE OF FORMULAS WITHOUT UNDERSTANDING OF THE FORMULAS

The indirect character of the higher types of arithmetical thinking has one very significant consequence It is possible to learn and to retain the formulas for the procedures which are implied in such thinking with

out passing through the experience of deriving the formulas Let us illustrate this statement by reference to a formula which is sometimes used in proving the correctness of processes in multiplication If we "cast out 9's," as it is called, in the multiplicand and in the multiplier and multiply the remainders in the two cases and again "cast out 9's, we get a number which corresponds exactly to the number left when the 9's have been cast out after adding the digits in the product Anyone who has been taught this rule can use it to prove multiplication and will ultimately cultivate confidence in the rule even though he cannot explain the rule and has no knowledge of the basis of the rule in the nature of the number series

There is so great a possibility of short circuiting mathematical operations in the fashion just described that a very large part of the study of arithmetic in the schools becomes a matter of accepting and learning rules rather than a matter of verifying processes Very few people have actually verified $11 \times 11 = 121$, and still fewer are clear as to the reason why inverting a fraction and multiplying is the same as dividing by the fraction which is inverted

It would be a highly interesting psychological inquiry to ascertain how far pupils who have studied arithmetic accept unverified rules and how far they have the more complete type of knowledge that gives them a thorough comprehension of the processes by which formulas are derived

LACK OF ANALYSIS IN TEST SCORES

Reverting to a matter which has been mentioned before, we note that the ordinary test in arithmetic makes no distinction whatever between mere memory of formulas and comprehension of the processes involved in arriving at formulas Indeed in some instances it is probably true that, so far as fluency in certain types of operations is concerned, the person who knows the rules thoroughly but has little comprehension of the processes makes better scores on tests than does the individual who has the more fundamental form of knowledge

UNDERSTANDING ESSENTIAL TO PROGRESS

The crucial test of complete knowledge comes when one must use a process of a lower level in order to understand a process of a higher type Anyone who knows only the rules of addition and does not understand the meaning of the process will never be able to understand the meaning of positive and negative quantities The person who knows merely the rules of pointing off decimals in division is very likely to become confused in many situations which he encounters

COMPLETE TRAINING INVOLVES UNDERSTANDING

It is evident on the grounds explained that instruction in arithmetic must combine in a proper way methods of manipulating numbers and acquisition of the meaning of the increasingly abstract ideas which are involved in such manipulation The teacher must teach pupils to add and should also make certain that there is developed an adequate under standing of what addition means

There are two ways in which a pupil may acquire a knowledge of what addition really means He may be brought into contact with a great many cases of addition and, by a gradual process of abstraction and generalization, arrive at an understanding of the process The alternative is that the teacher shall give explicit instruction in the meaning of the process of addition and thus hasten the arrival of the pupil at an understanding of the general idea

Evidently, the latter is the more economical procedure and the one which is to be advocated We are led, therefore, to see the importance of an analysis of the process of addition, we realize the pedagogical import of the lessons taught by the earlier chapters Teachers must be persuaded that addition is a process of combining or putting together and, as such, can be explained in general terms It is also subject to certain general laws This means that the person performing additions must understand that there is a definite relation between the numerical out come of addition and the quantities of the addends which are combined Neither the notion of combination nor the idea that addition follows pre cise laws can be omitted from the pupil s training if be is to be fully acquainted with arithmetic

THE TEACHING OF TWO ASPECTS OF ADDITION

Of the two phases of the addition idea that which is to be described as the notion of precision in thinking is by far the most novel to the pupil It is fairly easy to get the idea of combination but it requires a great deal of experimentation and systematization of the facts to be certain that one can add with precision A generation ago arithmetic was regarded as training in numerical precision Long drills were common in the schools on the purely numerical side It was assumed in earlier days that, if the school cultivated the idea of precision, the emergencies of life and a very little teaching would provide opportunity and adequate mo tives for the development of the general idea of combination Of late, there seems to be some relaxation of emphasis on precision and a dis position to give major attention to finding and displaying to pupils all

the different kinds of situations in which addition combinations are appropriate This latter form of emphasis on inductive methods is hazardous If the training in analysis of concrete situations is carried far enough it is doubtless possible to arrive in the long run, as the race has, at a full understanding of the number system and of the principles of precise thinking If only a limited number of practical situations are dealt with, the comprehension of the methods of being precise may be very incomplete because the practical situations may not contain enough cases of numbers to give the child an insight into the underlying principles of the number system and its true significance as a means of securing precision

The efficient teaching of addition requires explicit instruction in both phases of the process The lesson which educational psychology has for the teacher is that neither the descriptive accounts of situations on the one hand nor the rules of number combination on the other taken alone are adequate to a complete understanding of the chief lesson which arithmetic should teach namely, the lesson that precision in thought and action can be secured through the use of number

TEACHING THE OTHER FUNDAMENTAL OPERATIONS

The other forms of number combination and partition namely, the processes of subtraction multiplication and division were fully described and contrasted with addition in the immediately preceding chapter In the case of each of these forms of arithmetical manipulation it is necessary, as it is in the case of addition, to give attention to the number processes involved, to the situations in which the various types of combination or partition are appropriate, and to the general idea of precision In every case the teacher must attempt to induce in the mind of the pupil the development of general ideas As soon as the general ideas of the various types involved in the different kinds of combination are established and the results of various number groupings are known the pupil will have a true understanding of the particular section of arithmetic under discussion and will be in a position to appreciate more fully than before the broader general idea namely, the idea of precision

When the numbers involved in problems in arithmetic grow large and it becomes necessary to carry on the processes of combination with more than two digits new psychological factors are introduced The experiments reported in this monograph do not cover these more elaborate processes A recent study by Professor Buswell[1] shows that there are a

[1] G T Buswell with the co-operation of Lenore John *Diagnostic Studies in Arithmetic* Supplementary Educational Monographs No 30, Chicago Department of Education University of Chicago 1926

that is, the matter of curriculum construction Everywhere in the United States school systems are engaged in reorganizing the curriculum There is a tendency in many quarters to think of arithmetic as one of the subjects which can be drastically reduced It is sometimes urged that arithmetic be cut down to the point where only its practical applications will be included in the curriculum The conclusions to which the study reported in this monograph lead are diametrically opposed to the doctrine that arithmetic should be reduced to a few exercises in practical calculation If the experiments which have been reported prove anything they prove that the general ideas which are developed through contact with numbers can be cultivated in the individual only through a broad acquaintance with the properties of a highly perfected number system To eliminate number instruction from the schools or to give it only a minor place would be to suppress one of the most significant general ideas that the race has evolved To reduce arithmetic to a few practical applications would be to neglect the general idea of precise thinking on which our mechanical and scientific civilization rests

The issue between two fundamentally different views with regard to the curriculum which are now before the school people of this country is nowhere clearer than in the sphere of number training It is the issue between the view that the *duty of the school is the cultivation of comprehensive general ideas* and the view that the sole duty of the school is to train pupils in relatively trivial particular skills There can be no doubt that powerful influences are at work in the pedagogical world to reduce all training to the cultivation of practical and particular skills Fortunately, *the mind of man is so organized that it generalizes* Even if all the curriculum makers resolve to train nothing but particular abilities pupils will generalize and will continue to do what the race has done throughout its history, that is, abstract from particular situations those aspects which are most universal Some children will acquire the general idea of mathematical exactness no matter how far curriculum makers go in running counter to human history

THE PLACE OF GENERAL NUMBER IDEAS IN MODERN CIVILIZATION

It will certainly be to the advantage of the schools if the demand for a reduction of arithmetic is met by a vigorous counter demand for a more rational understanding of the place of number in the world of ideas and of its importance to civilization Curriculum makers should be urged to recognize the fact that the curriculum is made for the purpose of training *minds, not for the purpose of reflecting the immediate needs of practical*

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living. There are psychological requirements which must be met in curriculum-making, and these are infinitely more important for the future of the race than are the practical adjustments of trade and industry. When the Arabic numerals came into Europe, they found many skills highly developed, but they found little exact science In the short span of time which is marked by the lapse of less than four centuries, the industrial revolution has substituted machines, which are products of precision in thinking, for human hands, and exact science for blind trial The achievements of the industrial revolution would have been utterly impossible without number It is no trivial assault on the intellectual institutions of the race that is made by those who would dispense with mathematical education or reduce it to a bare minimum

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